How to insulate a pipe?

Fabien Caubet[*], Carlos Conca[†], Marc Dambrine[‡] and Rodrigo Zelada[§]

March 20, 2025

Abstract

This paper focuses on the problem of finding the optimal distribution of a thermal insulator around a pipe. We consider the framework of one fluid inside a pipe of thin width which is surrounded by a thermal insulator. We use an asymptotic model to avoid dealing with the thin layer, leading to non-standard transmission conditions which involve discontinuities at the interface and second order tangential derivatives. We thus consider the shape optimization problem that aims to minimize the heat flux outside an insulator with a given volume. Then we characterize the shape derivative of the objective functional and perform 3D numerical simulations using the level set evolution method.

Keywords: Shape optimization, asymptotic model, Ventcel conditions, transmission conditions, Navier-Stokes equations, thermal insulation

AMS Classification: 49Q10, 35Q79, 80M50, 76D55

1 Introduction and setting of the problem

1.1 Motivations

In our daily lives, reducing heat loss is of great importance from an ecological perspective. Indeed this problem appears in various contexts and applications such as hot water pipes, buildings, or electric kettles, for example. A typical question involves optimizing the thermal insulation around a pipe containing hot water, subject to a volume constraint on the insulator. Insulation problems have been studied for a long time. Here are a few recent advances (see, e.g., [8, 9]). In particular, in [8], two thermal insulation problems were addressed by parametrizing the insulating material by means of the tangential and normal coordinates on the boundary of the hot body (not a fluid) and minimizing with respect to the variable thickness: it was shown that under certain conditions, when the hot body is inside a ball, then the optimal insulator is a ball. Regarding the numerics, in [28], different configurations were compared and a heuristic was proposed to optimize a polygon satisfying certain geometrical constraints, providing insights into how the insulator should be configured.

In this work, we aim to provide practical numerical solutions of the optimal insulator problem. We therefore first perform a theoretical sensitivity analysis of the problem of insulating a pipe containing a hot fluid and then implement a descent method -here the null-space algorithm [21]- using the level set framework (see, e.g., [4]).

[*]Université de Pau et des Pays de l'Adour, E2S UPPA, CNRS, LMAP, UMR 5142, 64000 Pau, France. fabien.caubet@univ-pau.fr

[†]cconca@dim.uchile.cl

[‡]marc.dambrine@univ-pau.fr

[§]rodrigo.zelada-mancini@univ-pau.fr

In many practical applications, the thickness of the pipe is very small compared to its length and also small compared to the insulator thickness (see Figure 1, where the red part corresponds to the pipe thickness). For obvious computational reasons, one would preferably avoid to mesh it.

Figure 1: The thermal insulation of a pipe (photo by Sönke Kraft aka Arnulf zu Linden on commons.wikimedia.org).

One possibility would be to ignore it. In this work, we propose to take it into account by means of an interface condition written on the edge of the domain occupied by the fluid, obtained by an asymptotic model of order one with respect to the small parameter (i.e. the ratio between this thickness and the length of the pipe). The novelties of this work are: first, the model takes into account the motion of the fluid and thus a convection term appears in the heat equation in the fluid zone; second, the pipe is considered through an interface term rather than with the usual model of insulation (see [8, 9, 28]).

We emphasise that the fluid and the convection in the domain Ω_1 model the physics of the problem but that Ω_1 will not be optimised. In particular it is important to note that the transmission conditions that we will impose on the boundary of Ω_1 will thus be imposed on a fixed boundary, which will not be subject to shape optimisation. Indeed only the boundary of the insulating phase is optimized, which lies "far" from these complicated feature. This non-trivial difficulty will be the subject of further work.

1.2 The physical context

As previously mentioned, we consider in this work a given pipe, with a known geometry. The thickness of the wall of the pipe is very small compared to the length of the pipe. Moreover, we assume that this pipe is surrounded by an insulator.

More precisely, the problem can be stated as follows. Let $\epsilon > 0$. Let Ω be an open bounded connected domain of $\mathbb{R}^d$, $(d = 2, 3)$, divided into three open bounded subdomains $\Omega_1^\epsilon, \Omega_{\mathrm{m}}^\epsilon, \Omega_2^\epsilon$, where $\Omega_{\mathrm{m}}^\epsilon$ has constant thickness ϵ and separates Ω_1^ϵ from Ω_2^ϵ (see Figure 2). The subset Ω_1^ϵ is the part physically occupied by the fluid, $\Omega_{\mathrm{m}}^\epsilon$ corresponds to the wall of the pipe, and finally Ω_2^ϵ

corresponds to the insulating material we will be looking to place to reduce heat loss. The interfaces between the domains are denoted by $\Gamma_i^\epsilon := \overline{\partial\Omega_i^\epsilon} \cap \overline{\partial\Omega_\mathrm{m}^\epsilon}$, $i = 1, 2$, and we also assume that $\Gamma_1^\epsilon \cap \Gamma_2^\epsilon = \emptyset$. We define $\Gamma^\epsilon := \Gamma_1^\epsilon \cup \Gamma_2^\epsilon$. Throughout this article, we denote by $\boldsymbol{n}$ the outer unit normal to $\partial\Omega$. Moreover, particular attention should be paid to the normal used on Γ_1^ϵ and Γ_2^ϵ, where we have chosen to orient the normal exterior to Ω_1^ϵ and interior to Ω_2^ϵ, respectively, i.e., we define $\boldsymbol{n} = \boldsymbol{n}_1^\epsilon = -\boldsymbol{n}_\mathrm{m}^\epsilon$ on Γ_1^ϵ and $\boldsymbol{n} = -\boldsymbol{n}_2^\epsilon = \boldsymbol{n}_\mathrm{m}^\epsilon$ on Γ_2^ϵ (where, for $i = 1, 2, \mathrm{m}$, $\boldsymbol{n}_i^\epsilon$ denote the exterior unit normal to Ω_i^ϵ).

As the pipe may have a complex geometry, the flow of the heat transfer fluid is described by the stationary incompressible Navier-Stokes equations (rather than the Stokes equations). We precise that we consider here the steady state case and we denote by $\boldsymbol{u}^\epsilon$ the velocity of the fluid and p^ϵ the pressure. Let $\nu, \rho > 0$ be the kinematic viscosity and the material density, respectively. The boundary of the fluid region Ω_1^ϵ is assumed to be composed of three disjoint regions: $\partial\Omega_1^\epsilon =: \Gamma_\mathrm{D}^\epsilon \cup \Gamma_1^\epsilon \cup \Gamma_\mathrm{N}^\epsilon$, where $\Gamma_\mathrm{D}^\epsilon$ is the input of the fluid with a given velocity (non homogeneous Dirichlet boundary condition) and where the full traction vector is prescribed on $\Gamma_\mathrm{N}^\epsilon$. The classical no-slip condition $\boldsymbol{u}^\epsilon = 0$ is imposed on Γ_1^ϵ. To summarize, the motion of the fluid is described by the following equations:

$$\begin{cases} -\nu\Delta\boldsymbol{u}^\epsilon + (\nabla\boldsymbol{u}^\epsilon)\boldsymbol{u}^\epsilon + \dfrac{1}{\rho}\nabla p^\epsilon &= 0 \quad \text{in } \Omega_1^\epsilon, \\ \mathrm{div}(\boldsymbol{u}^\epsilon) &= 0 \quad \text{in } \Omega_1^\epsilon, \\ \boldsymbol{u}^\epsilon &= \boldsymbol{u}_\mathrm{D} \quad \text{on } \Gamma_\mathrm{D}^\epsilon, \\ \sigma(\boldsymbol{u}^\epsilon, p^\epsilon)\boldsymbol{n} &= 0 \quad \text{on } \Gamma_\mathrm{N}^\epsilon, \\ \boldsymbol{u}^\epsilon &= 0 \quad \text{on } \Gamma_1^\epsilon, \end{cases} \tag{1.1}$$

where $\boldsymbol{u}_\mathrm{D}$ is a given inlet velocity, and where $\sigma(\boldsymbol{u}, p)$ is the fluid stress tensor defined by

$$\sigma(\boldsymbol{u}, p) := 2\nu\varepsilon(\boldsymbol{u}) - \frac{p}{\rho}\mathrm{I},$$

with $\varepsilon(\boldsymbol{u}) := \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^t\right)$ the symmetric gradient and I the identity matrix, and where the superscript ^{t} denotes the transpose matrix.

Remark 1.1. *In practice, we work with the unit pressure head* $\bar{p} = \dfrac{p}{\rho}$.

Concerning the temperature, we consider the stationary heat equation in the whole domain Ω^ϵ with a Fourier-Robin condition on the outer boundary of the insulating material stating that the heat flux there is proportional to the gap of temperatures between the considered domain and the outer medium with a given rate $\alpha > 0$. Notice that the temperature field inside the pipe Ω_1^ϵ is determined in terms of the velocity $\boldsymbol{u}^\epsilon$ through a convection–diffusion equation. It is convenient to decompose the temperature field into

$$\mathsf{T}^\epsilon = \mathsf{T}_1^\epsilon \mathbb{1}_{\Omega_1^\epsilon} + \mathsf{T}_\mathrm{m}^\epsilon \mathbb{1}_{\Omega_\mathrm{m}^\epsilon} + \mathsf{T}_2^\epsilon \mathbb{1}_{\Omega_2^\epsilon},$$

where T^ϵ is the solution of the stationary convection-diffusion equation in Ω^ϵ and where T_i^ϵ is its restriction to Ω_i^ϵ, for $i = 1, 2, \mathrm{m}$. Here $\mathbb{1}$ denotes the indicator function of a domain. The physical parameters are the given thermal diffusivities $\kappa_1, \kappa_2, \kappa_\mathrm{m}$ that are assumed to be positive numbers. The boundary of Ω_2^ϵ is the disjoint union $\partial\Omega_2^\epsilon =: \Gamma_2^\epsilon \cup \Gamma_\mathrm{R}^\epsilon \cup \Gamma_e^\epsilon$ and the boundary of $\Omega_\mathrm{m}^\epsilon$ is given by $\partial\Omega_\mathrm{m}^\epsilon =: \Gamma^\epsilon \cup \Gamma_{\mathrm{m,N}}^\epsilon$. Figures 2 and 3a illustrate the geometry of our problem. On the Dirichlet part $\Gamma_\mathrm{D}^\epsilon$, a given temperature is imposed, and the previously mentioned Fourier-Robin condition is imposed on $\Gamma_\mathrm{R}^\epsilon$. Moreover, we impose Neumann boundary conditions on $\Gamma_{\mathrm{m,N}}^\epsilon \cup \Gamma_\mathrm{N}^\epsilon \cup \Gamma_\mathrm{e}^\epsilon$. Finally,

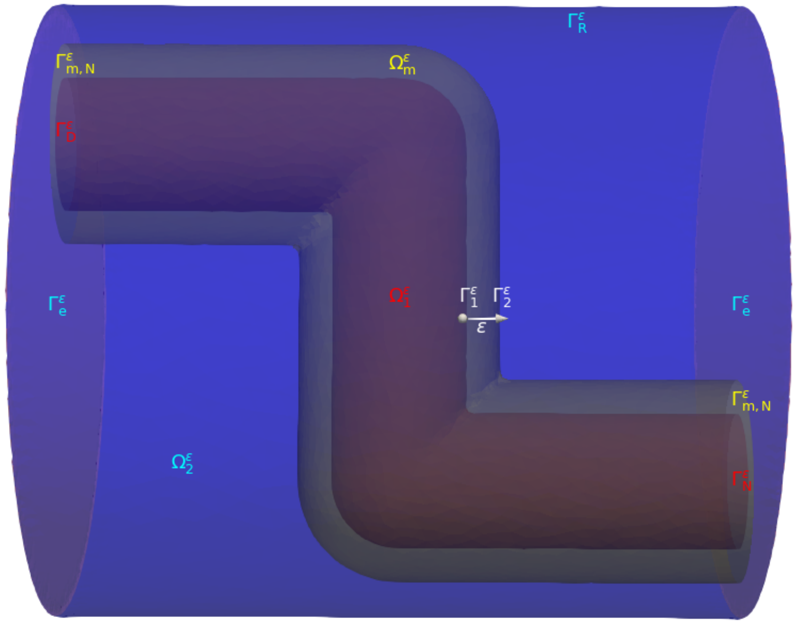


Figure 2: Configuration of our 3D thermal insulation problem in Ω divided by $\Omega_1^\epsilon, \Omega_{\rm m}^\epsilon$ and Ω_2^ϵ.

classical transmission conditions are assumed on Γ^ϵ and we obtain the following system:

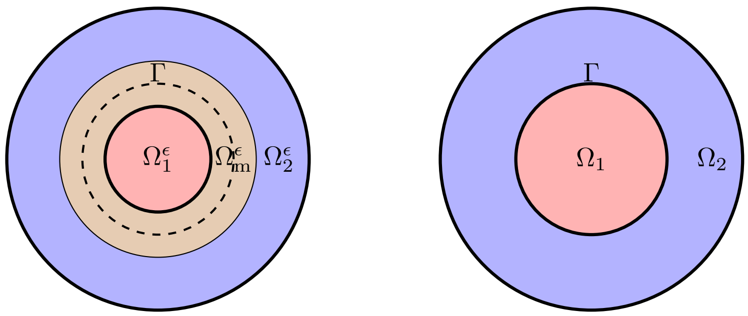
$$\left\{\begin{array}{rcll}
-\operatorname{div}(\kappa_1 \nabla \mathsf{T}_1^\epsilon) + \boldsymbol{u}^\epsilon \cdot \nabla \mathsf{T}_1^\epsilon &=& 0 & \text{in } \Omega_1^\epsilon,\\
-\operatorname{div}(\kappa_{\rm m} \nabla \mathsf{T}_{\rm m}^\epsilon) &=& 0 & \text{in } \Omega_{\rm m}^\epsilon,\\
-\operatorname{div}(\kappa_2 \nabla \mathsf{T}_2^\epsilon) &=& 0 & \text{in } \Omega_2^\epsilon,\\
\mathsf{T}_1^\epsilon &=& \mathsf{T}_{\rm D} & \text{on } \Gamma_{\rm D}^\epsilon,\\
\kappa_1 \dfrac{\partial \mathsf{T}_1^\epsilon}{\partial \boldsymbol{n}} &=& 0 & \text{on } \Gamma_{\rm N}^\epsilon,\\
\kappa_m \dfrac{\partial \mathsf{T}_m^\epsilon}{\partial \boldsymbol{n}} &=& 0 & \text{on } \Gamma_{\rm m,N}^\epsilon,\\
\kappa_2 \dfrac{\partial \mathsf{T}_2^\epsilon}{\partial \boldsymbol{n}} &=& 0 & \text{on } \Gamma_{\rm e}^\epsilon,\\
\kappa_2 \dfrac{\partial \mathsf{T}_2^\epsilon}{\partial \boldsymbol{n}} + \alpha \mathsf{T}_2^\epsilon &=& \alpha \mathsf{T}_{\rm ext} & \text{on } \Gamma_{\rm R}^\epsilon,\\
\mathsf{T}_1^\epsilon &=& \mathsf{T}_{\rm m}^\epsilon & \text{on } \Gamma_1^\epsilon,\\
\mathsf{T}_2^\epsilon &=& \mathsf{T}_{\rm m}^\epsilon & \text{on } \Gamma_2^\epsilon,\\
\kappa_1 \dfrac{\partial \mathsf{T}_1^\epsilon}{\partial \boldsymbol{n}} &=& \kappa_{\rm m} \dfrac{\partial \mathsf{T}_{\rm m}^\epsilon}{\partial \boldsymbol{n}} & \text{on } \Gamma_1^\epsilon,\\
\kappa_2 \dfrac{\partial \mathsf{T}_2^\epsilon}{\partial \boldsymbol{n}} &=& \kappa_{\rm m} \dfrac{\partial \mathsf{T}_{\rm m}^\epsilon}{\partial \boldsymbol{n}} & \text{on } \Gamma_2^\epsilon,
\end{array}\right. \tag{1.2}$$

where $\mathsf{T}_{\rm D}$ is a given input temperature and $\mathsf{T}_{\rm ext}$ is the given exterior temperature.

1.3 Setting of the approximate and of the shape optimization problems

As exposed above, the actual configuration involves three disjoint regions: the fluid, the wall of the pipe, and the insulator. However, the wall thickness is very small compared to the other dimensions, and keeping this wall in a numerical model requires the use of very refined and therefore very expensive meshes, especially in dimension three, to compute the temperature field.

The approximate domains. We therefore propose, in a classical way since the work of Enquist and Nedelec [18], to forget this zone in the geometrical description of the problem, but to take into account its impact on the thermal properties through artificial transmission conditions at the new fluid-insulator interface. We thus obtain, at the cost of a systematic model error, an approximate solution whose calculation is much less costly since it only requires a much coarser mesh (adapted to the internal diameter of the pipe and no longer to the wall thickness). Our idea is to use this inexpensive approximate model to optimize the shape of the insulation around the pipe. Figure 3 illustrates this geometric approximation.



(a) The real domain Ω^ϵ. (b) The approximate domain.

Figure 3: Approximation of the domain from three to two layers (cross-section view)

Thus, in the following, we only consider a two layer domain by doing $\epsilon \to 0$: one occupied by the fluid (denoted by Ω_1) and one by the insulator (denoted by Ω_2), separated by the interface $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$ that we assume to have non-zero measure in $\mathbb{R}^{d-1}$ and to be $\mathcal{C}^1$ (see Figure 3b). Roughly speaking, Ω_i is the limit of Ω_i^ϵ when $\epsilon \to 0$, $i = 1, 2$, $\Omega_{\mathrm{m}}^\epsilon$ disappears and Γ coincides with the mid-surface between Γ_1^ϵ and Γ_2^ϵ. We then define $\Omega := \Omega_1 \cup \Omega_2$. In a similar way to above, $\partial\Omega_1$ and $\partial\Omega_2$ are respectively decomposed as $\partial\Omega_1 =: \Gamma_{\mathrm{D}} \cup \Gamma \cup \Gamma_{\mathrm{N}}$ and $\partial\Omega_2 =: \Gamma \cup \Gamma_{\mathrm{R}} \cup \Gamma_{\mathrm{e}}$. Moreover, we assume that $\overline{\Gamma} \cap \overline{\Gamma_{\mathrm{D}}} \neq \emptyset$ and $\overline{\Gamma} \cap \overline{\Gamma_{\mathrm{N}}} \neq \emptyset$. Finally, we assume in the following that the normals to Γ_{N} and Γ_{D}, $\boldsymbol{n}_{\Gamma_{\mathrm{N}}}$ and $\boldsymbol{n}_{\Gamma_{\mathrm{D}}}$, are unit tangent vectors to Γ on $\partial\Gamma \cap \partial\Gamma_{\mathrm{N}}$ and $\partial\Gamma \cap \partial\Gamma_{\mathrm{D}}$. Figure 4 illustrates our configuration.

The approximate equations. We can now specify the boundary values problems that we will consider in the following. Concerning the fluid, the system is similar since the equations are not affected by this reduction of the domain. Hence, we consider

$$\left\{\begin{aligned} -\nu\Delta\boldsymbol{u} + (\nabla\boldsymbol{u})\boldsymbol{u} + \frac{1}{\rho}\nabla p &= 0 && \text{in } \Omega_1,\\ \operatorname{div}(\boldsymbol{u}) &= 0 && \text{in } \Omega_1,\\ \boldsymbol{u} &= \boldsymbol{u}_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}},\\ \sigma(\boldsymbol{u},p)\boldsymbol{n} &= 0 && \text{on } \Gamma_{\mathrm{N}},\\ \boldsymbol{u} &= 0 && \text{on } \Gamma, \end{aligned}\right. \tag{1.3}$$

where $\boldsymbol{u}_{\mathrm{D}} \in \mathrm{H}_{00}^{1/2}(\Gamma_{\mathrm{D}})^d := \{\boldsymbol{v}|_{\Gamma_{\mathrm{D}}}, \boldsymbol{v} \in \mathrm{H}^1(\Omega_1)^d, \boldsymbol{v}|_{\partial\Omega_1\setminus\overline{\Gamma_{\mathrm{D}}}} = 0\}$ is the given inlet velocity. It is well-known that theses stationary incompressible Navier-Stokes equations are well-posed if ν is large enough (see, e.g., [7]), which we will assume in the remainder of this work: then it admits a unique solution $(\boldsymbol{u}, p) \in \mathrm{H}^1(\Omega_1)^d \times \mathrm{L}^2(\Omega_1)$.

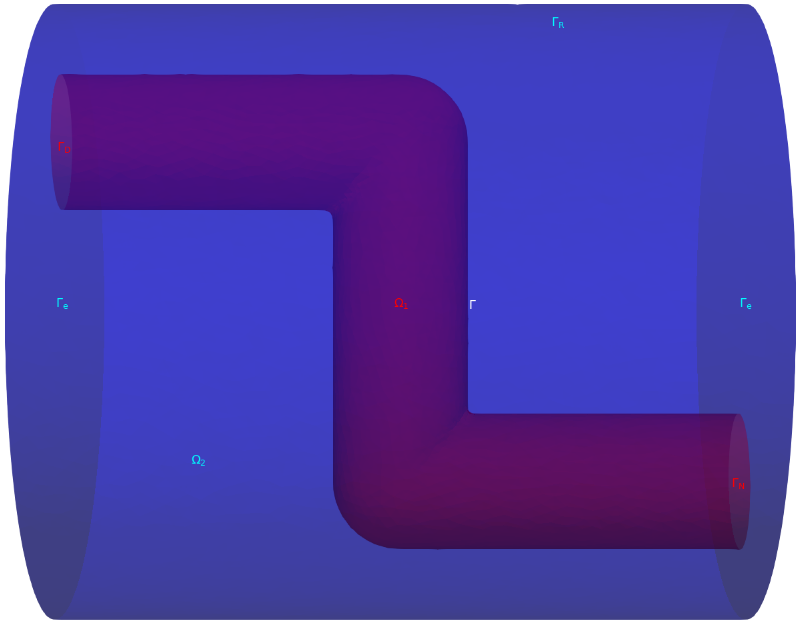


Figure 4: Configuration of the 3D thermal insulation problem in Ω divided by Ω_1 and Ω_2.

Remark 1.2. *As previously mentioned, the shape Ω_1 will not be subject to shape optimization and is therefore fixed. It is then important to note that, in practise, it is solved once and for all throughout the optimization process.*

As far as the thermal equations are concerned, the omission of the wall of the pipe has a significant impact on the equations. We use the approach introduced in [18] based on now classic asymptotical techniques. This approach and technical implementation are explained in detail and pedagogically in Vial's thesis [29]. We also mention the work [14] dealing with generalized boundary conditions for an interface problem. As previously, we use the following decomposition:

$$\mathsf{T} = \mathsf{T}_1 \mathbb{1}_{\Omega_1} + \mathsf{T}_2 \mathbb{1}_{\Omega_2}, \kappa = \kappa_1 \mathbb{1}_{\Omega_1} + \kappa_2 \mathbb{1}_{\Omega_2}.$$

Then doing similar computations to what we have done in [12], when ϵ is sufficiently small, the following approximate problem of order one is obtained:

$$
\begin{cases}
-\operatorname{div}(\kappa_1\nabla\mathsf{T}_1)+\boldsymbol{u}\cdot\nabla\mathsf{T}_1 &= 0 & \text{in } \Omega_1,\\
-\operatorname{div}(\kappa_2\nabla\mathsf{T}_2) &= 0 & \text{in } \Omega_2,\\
\mathsf{T}_1 &= \mathsf{T}_{\mathrm{D}} & \text{on } \Gamma_{\mathrm{D}},\\
\kappa_1\dfrac{\partial\mathsf{T}_1}{\partial\boldsymbol{n}} &= 0 & \text{on } \Gamma_{\mathrm{N}},\\
\kappa_2\dfrac{\partial\mathsf{T}_2}{\partial\boldsymbol{n}} &= 0 & \text{on } \Gamma_{\mathrm{e}},\\
\kappa_2\dfrac{\partial\mathsf{T}_2}{\partial\boldsymbol{n}}+\alpha\mathsf{T}_2 &= \alpha\mathsf{T}_{\mathrm{ext}} & \text{on } \Gamma_{\mathrm{R}},\\
\left\langle\kappa\dfrac{\partial\mathsf{T}}{\partial\boldsymbol{n}}\right\rangle &= -\kappa_{\mathrm{m}}\epsilon^{-1}\,[\mathsf{T}] & \text{on } \Gamma,\\
\left[\kappa\dfrac{\partial\mathsf{T}}{\partial\boldsymbol{n}}\right] &= \epsilon\operatorname{div}_\tau(\kappa_{\mathrm{m}}\nabla_\tau\langle\mathsf{T}\rangle)-\kappa_{\mathrm{m}}H[\mathsf{T}] & \text{on } \Gamma,\\
\kappa_i\dfrac{\partial\mathsf{T}_i}{\partial\boldsymbol{n}} &= 0 & \text{on } \partial\Gamma, i=1,2,
\end{cases}
\tag{1.4}
$$

where $\mathsf{T}_{\mathrm{D}} \in \mathrm{H}^{1/2}(\Gamma_{\mathrm{D}})$ is the given exterior temperature, and where $\boldsymbol{u}$ solves the Navier-Stokes system (1.3). In the previous equations, div_τ and ∇_τ are, respectively, the tangential divergence and gradient operator, and H is the mean curvature of Γ. Since the jumps are not zero, we need to choose an orientation for the normal: let $\boldsymbol{n}$ the outer unit normal at Γ oriented towards Ω_2, this is $\boldsymbol{n} := \boldsymbol{n}_1 = -\boldsymbol{n}_2$ at Γ, where $\boldsymbol{n}_i$ is the exterior normal to Ω_i. The jump and mean across the interface Γ are defined, for a function ϕ, as

$$
[\phi] := \phi_1 - \phi_2 \qquad \text{and} \qquad \langle\phi\rangle := \frac{1}{2}\left(\phi_1+\phi_2\right).
$$

The well-posedness of this system is proved below (see Theorem 2.2). Roughly speaking, this asymptotic model takes into account the physics in the volume $\Omega^\epsilon_{\mathrm{m}}$ through the transmission conditions on Γ in which appears the tangential (surface) Laplacian $\Delta_\tau = \operatorname{div}_\tau(\operatorname{grad}_\tau)$, also known as the Laplace-Beltrami operator. These transmission conditions are generally referred to as Ventcel conditions.

Remark 1.3. *We emphasize here that the condition on $\partial\Gamma$ come from the asymptotic expansion taking into account the homogeneous Neumann condition imposed on $\Gamma^\epsilon_{\mathrm{m,N}}$ in* (1.2) *and the zero jumps imposed on Γ^ϵ_i for $i = 1, 2$. Indeed $\kappa_i\frac{\partial\mathsf{T}^\epsilon_i}{\partial\boldsymbol{n}} = 0$ on $\partial\Gamma^\epsilon_i$ since $\mathsf{T}^\epsilon_i = \mathsf{T}^\epsilon_{\mathrm{m}}$ on Γ^ϵ_i and $\kappa_{\mathrm{m}}\frac{\partial\mathsf{T}^\epsilon_{\mathrm{m}}}{\partial\boldsymbol{n}} = 0$ on $\Gamma^\epsilon_{\mathrm{m,N}}$.*

Remark 1.4. *It should be pointed out that the solution T of the approximate problem obviously depends on the parameter ϵ. It is then necessary to justify that the model error committed by using T instead of T^ϵ is of order ϵ^2 in H^1 norm. This is a little technical, as the two functions are not defined on the same functional space: $\mathsf{T}^\epsilon \in \mathrm{H}^1(\Omega)$, while T belongs to a broken Sobolev space and then, since it is discontinuous, it does not belong to H^1 globally in Ω (it simply belongs to H^1 in each subdomain Ω_i, $i = 1, 2$). Nevertheless, it is a classic result. That is not the point of our work, so we refer the reader to [14] for an example where a similar result is demonstrated and just comment on it. Obviously, the systematic error committed by approximating T^ϵ by T is weaker than that committed by setting $\epsilon = 0$ in* (1.4)*, that is a classical two-phase continuous problem.*

Strictly speaking, this dependence of the solution of the approximate problem on the small parameter ϵ should be explicitly mentioned. As this would make the notations much heavier, we chose to simply denote T the solution of the previous approximated problem.

The shape optimization problem. We can now set out the main question that we are going to study in this work: *given a pipe with a known fixed geometry and a given quantity of insulation, how should the insulation be positioned to minimize heat loss to the outside world, whose temperature is known?* In other words, the domain Ω_1 being fixed, we are looking for a domain Ω_2 of prescribed volume so that the heat flux across the interface with the outside, i.e. Γ_{R}, is as small as possible. We therefore define the criterion J by

$$J(\Omega_2) := \int_{\Gamma_{\mathrm{R}}} \left(\kappa_2 \frac{\partial \mathsf{T}_2}{\partial \boldsymbol{n}} \right)^2 \mathrm{d}s = \int_{\Gamma_{\mathrm{R}}} \alpha^2 (\mathsf{T}_2 - \mathsf{T}_{\mathrm{ext}})^2 \, \mathrm{d}s, \tag{1.5}$$

where the temperature T solves the approximate convection-diffusion problem (1.4).

We thus consider the following shape optimization problem: *given a prescribed volume $V_0 > 0$, minimize J under the constraint*

$$G(\Omega_2) = 0, \qquad \text{where } G(\Omega_2) := \int_{\Omega_2} \mathrm{d}x - V_0.$$

The fundamental questions of the existence of optimal domains and their regularity have been studied in the work of Bucur *et al.* [9] in a simplified setting (no pipe wall and no fluid circulation just a heated body). The study of these questions is not the topic of the present work. We will here focus on the numerical computation of such a solution, and to this end prove the differentiability of the shape functionals at stake and compute the shape derivatives in this framework.

Remark 1.5. *We can notice that the real heat loss is the functional*

$$Q(\Omega_2) := \int_{\Gamma_{\mathrm{R}}} -\kappa_2 \frac{\partial \mathsf{T}_2}{\partial \boldsymbol{n}} \, \mathrm{d}s = \int_{\Gamma_{\mathrm{R}}} \alpha (\mathsf{T}_2 - \mathsf{T}_{\mathrm{ext}}) \, \mathrm{d}s,$$

but we prefer the square version in order to have positive values.

2 Main results

In this section, we state the main results of our work. All the proofs are detailed in the following section.

2.1 Functional spaces and well posedness of the approximate problem (1.4)

We consider the following affine spaces associated to the non-homogeneous Dirichlet boundary data $\boldsymbol{u}_{\mathrm{D}} \in \mathrm{H}^{1/2}_{00}(\Gamma_{\mathrm{D}})^d$ and $\mathsf{T}_{\mathrm{D}} \in \mathrm{H}^{1/2}(\Gamma_{\mathrm{D}})$:

$$\begin{aligned} \mathcal{V}_{\boldsymbol{u}_{\mathrm{D}}}(\Omega_1) &:= \{\boldsymbol{w} \in \mathrm{H}^1(\Omega_1)^d;\ \boldsymbol{w} = \boldsymbol{u}_{\mathrm{D}} \text{ on } \Gamma_{\mathrm{D}}, \boldsymbol{w} = 0 \text{ on } \Gamma\}, \\ \mathcal{H}_{\mathsf{T}_{\mathrm{D}}}(\Omega_1, \Omega_2) &:= \{\mathsf{S} = (\mathsf{S}_1, \mathsf{S}_2) \in \mathcal{H}^1(\Omega_1, \Omega_2);\ \mathsf{S}_1 = \mathsf{T}_{\mathrm{D}} \text{ on } \Gamma_{\mathrm{D}}\}, \end{aligned}$$

where,

$$\mathcal{H}^k(\Omega_1, \Omega_2) := \{\mathsf{S} = (\mathsf{S}_1, \mathsf{S}_2) \in \mathrm{H}^1(\Omega_1) \times \mathrm{H}^1(\Omega_2);\ \langle \mathsf{S} \rangle \in \mathrm{H}^1(\Gamma)\}, k = 1, 2.$$

The spaces $\mathcal{V}_0(\Omega_1)$ and $\mathcal{H}_0(\Omega_1, \Omega_2)$ are Hilbert spaces when they are equipped with the respective norms:

$$\|\boldsymbol{w}\|_{\mathcal{V}_0(\Omega_1)} := \|\boldsymbol{w}\|_{\mathrm{H}^1(\Omega_1)^d} \text{ and } \|\mathsf{S}\|_{\mathcal{H}_0(\Omega_1,\Omega_2)} := \left(\sum_{i=1}^{2} \|\nabla \mathsf{S}_i\|^2_{\mathrm{L}^2(\Omega_i)^d} + \|\nabla_\tau \langle \mathsf{S} \rangle\|^2_{\mathrm{L}^2(\Gamma)^d} + \|[\mathsf{S}]\|^2_{\mathrm{L}^2(\Gamma)} \right)^{1/2}.$$

Then the incompressible Navier-Stokes equations (1.3) have the following variational formulation

$$\left\{\begin{array}{l}\text{Find } (\boldsymbol{u},p)\in\mathcal{V}_{\boldsymbol{u}_{\mathrm{D}}}(\Omega_1)\times\mathrm{L}^2(\Omega_1) \text{ such that, for all } (\boldsymbol{w},r)\in\mathcal{V}_0(\Omega_1)\times\mathrm{L}^2(\Omega_1),\\ \displaystyle\int_{\Omega_1}\left(2\nu\varepsilon(\boldsymbol{u}):\varepsilon(\boldsymbol{w})+(\nabla\boldsymbol{u})\boldsymbol{u}\cdot\boldsymbol{w}-\frac{p}{\rho}\operatorname{div}(\boldsymbol{w})-\frac{r}{\rho}\operatorname{div}(\boldsymbol{u})\right)\mathrm{d}x=0.\end{array}\right. \tag{2.1}$$

As previously mentioned, we assume that the viscosity ν is large enough so that the problem (1.3) is well-posed: it has a unique weak solution $(\boldsymbol{u},p)\in\mathcal{V}_{\boldsymbol{u}_{\mathrm{D}}}(\Omega_1)\times\mathrm{L}^2(\Omega_1)$. For the remainder of this work, we assume that the velocity of the fluid at the outlet, i.e. at the boundary Γ_{N}, actually causes it to exit: there is no recirculation at the outlet. This assumption on the velocity is written as

$$\boldsymbol{u}\cdot\boldsymbol{n}\geq 0 \ \text{ on } \Gamma_{\mathrm{N}}. \tag{2.2}$$

For the temperature, the corresponding variational formulation of the approximate problem (1.4) is given by

$$\text{Find } \mathsf{T}\in\mathcal{H}_{\mathsf{T}_{\mathrm{D}}}(\Omega_1,\Omega_2) \ \text{ such that, for all } \mathsf{S}\in\mathcal{H}_0(\Omega_1,\Omega_2),\ a(\mathsf{T},\mathsf{S})=l(\mathsf{S}), \tag{2.3}$$

where the bilinear and linear forms are respectively

$$\begin{aligned} a(\mathsf{T},\mathsf{S}) \ &:= \ \sum_{i=1}^{2}\int_{\Omega_i}\kappa_i\nabla\mathsf{T}_i\cdot\nabla\mathsf{S}_i\,\mathrm{d}x+\int_{\Omega_1}\mathsf{S}_1\boldsymbol{u}\cdot\nabla\mathsf{T}_1\,\mathrm{d}x+\int_{\Gamma_{\mathrm{R}}}\alpha\mathsf{T}_2\mathsf{S}_2\,\mathrm{d}s\\ &\quad+\int_{\Gamma}\kappa_{\mathrm{m}}\left(\epsilon\nabla_\tau\langle\mathsf{T}\rangle\cdot\nabla_\tau\langle\mathsf{S}\rangle+H[\mathsf{T}]\langle\mathsf{S}\rangle+\frac{1}{\epsilon}[\mathsf{T}][\mathsf{S}]\right)\mathrm{d}s,\\ l(\mathsf{S}) \ &:= \ \int_{\Gamma_{\mathrm{R}}}\alpha\mathsf{T}_{\mathrm{ext}}\mathsf{S}_2\,\mathrm{d}x. \end{aligned}$$

Remark 2.1. *Here we highlight an important point in obtaining the previous variational formulation. Let T be the strong solution of* (1.4) *that we suppose $\mathcal{H}^2(\Omega_1,\Omega_2)$ and $\mathsf{S}\in\mathcal{H}_0(\Omega_1,\Omega_2)$. Using Green's formula on the boundary Γ (see, e.g., [27, Proposition 2.58]), we obtain*

$$\int_\Gamma-\operatorname{div}_\tau(\nabla_\tau\langle\mathsf{T}\rangle)\langle\mathsf{S}\rangle\,\mathrm{d}s=\int_\Gamma\nabla_\tau\langle\mathsf{T}\rangle\cdot\langle S\rangle\,\mathrm{d}s-\int_{\partial\Gamma}\langle\mathsf{S}\rangle\nabla_\tau\langle\mathsf{T}\rangle\cdot\boldsymbol{\tau}\,\mathrm{d}l,$$

where $\boldsymbol{\tau}$ is the unit tangent vector to Γ, normal to $\partial\Gamma$ and $\mathrm{d}l$ is the $(d-2)$ dimensional measure along $\partial\Gamma$. In our situation, τ corresponds to the normal to Γ_{D} on $\overline{\Gamma}\cap\overline{\Gamma}_{\mathrm{D}}$ and the normal to Γ_{N} on $\overline{\Gamma}\cap\overline{\Gamma}_{\mathrm{N}}$. Then the second term of the right hand-side of the previous formula vanishes since $\nabla_\tau\langle\mathsf{T}\rangle\cdot\boldsymbol{\tau}=\dfrac{\partial\langle\mathsf{T}\rangle}{\partial\boldsymbol{n}}=\left\langle\dfrac{\partial\mathsf{T}}{\partial\boldsymbol{n}}\right\rangle$ and since $\dfrac{\partial\mathsf{T}_i}{\partial\boldsymbol{n}}=0$ on $\partial\Gamma$, $i=1,2$ (which comes from the boundary conditions on $\partial\Gamma$ given in (1.4)*)*

Then the following result (proved in Section 3) claims that this problem is well-posed.

Theorem 2.2 (Well-posedness of the state equation for temperature)**.** *Assume that the exit condition* (2.2) *holds. There exists a positive real number ϵ_0 such that, if $0<\epsilon<\epsilon_0$, then the convection-diffusion problem* (2.3) *has a unique solution $\mathsf{T}\in\mathcal{H}_{\mathsf{T}_{\mathrm{D}}}(\Omega_1,\Omega_2)$.*

2.2 Shape sensitivity analysis

Now we aim to perform a shape sensitivity analysis and compute the shape derivative of the objective functional J given in (1.5). To do this, we rely on the Hadamard shape derivative (see [27, 23] among many). We suppose Ω_2 to be smooth enough (at least $\mathcal{C}^2$). The main idea is to perturb the domain Ω_2 (in particular, the free boundary Γ_{R}) using a vector deformation field $\boldsymbol{\theta} \in \mathcal{C}^{1,\infty}(\mathbb{R}^d)^d := \mathcal{C}^1 \cap \mathrm{W}^{1,\infty}(\mathbb{R}^d)^d$ with $\|\boldsymbol{\theta}\|_{\mathrm{W}^{1,\infty}(\mathbb{R}^d)^d} < 1$, this is,

$$\Omega_2^{\boldsymbol{\theta}} := (\boldsymbol{I} + \boldsymbol{\theta})\Omega_2.$$

We consider the following space of admissible deformations,

$$\Theta_{\mathrm{ad}} := \{\boldsymbol{\theta} \in \mathcal{C}^{1,\infty}(\mathbb{R}^d)^d; \quad \|\boldsymbol{\theta}\|_{\mathrm{W}^{1,\infty}(\mathbb{R}^d)^d} < 1, \quad \boldsymbol{\theta} = 0 \text{ in } \overline{\Omega_1}\}.$$

We first recall the definition of the notion of shape derivative of a shape functional in our context.

Definition 2.3. *The shape derivative of a function $J(\Omega_2)$ is defined as the Fréchet derivative at 0 of the map $\boldsymbol{\theta} \in \Theta_{\mathrm{ad}} \mapsto J(\Omega_2^{\boldsymbol{\theta}}) \in \mathbb{R}$. It is denoted by $J'(\Omega_2)$ and it is then given by*

$$J(\Omega_2^{\boldsymbol{\theta}}) = J(\Omega_2) + \langle J'(\Omega_2), \boldsymbol{\theta}\rangle + o(\boldsymbol{\theta}), \ \ \textit{with} \ \lim_{\boldsymbol{\theta}\to 0} \frac{o(\boldsymbol{\theta})}{\|\boldsymbol{\theta}\|_{\mathrm{W}^{1,\infty}(\Omega)^d}} = 0.$$

In the following, we introduce $\mathsf{T}_{\boldsymbol{\theta}} \in \mathcal{H}_{\mathsf{T}_{\mathrm{D}}}(\Omega_1, \Omega_2^{\boldsymbol{\theta}})$ the perturbed solution, i.e. the solution of approximate convection-diffusion Problem (1.4) defined on $\Omega_1 \cup \Omega_2^{\boldsymbol{\theta}}$ instead of $\Omega_1 \cup \Omega_2$.

Proposition 2.4 (Existence and characterization of the shape derivative)**.** *If $\mathsf{T}_{\mathrm{ext}} \in \mathrm{H}^2(\mathbb{R}^d)$, then there exists an extension $\tilde{\mathsf{T}}_{\boldsymbol{\theta}} \in \mathrm{H}^1(\mathbb{R}^d) \times \mathrm{H}^1(\mathbb{R}^d)$ of $\mathsf{T}_{\boldsymbol{\theta}}$ such that the mapping $\boldsymbol{\theta} \mapsto \tilde{\mathsf{T}}_{\boldsymbol{\theta}}$ from Θ_{ad} into $\mathrm{L}^2(\mathbb{R}^d) \times \mathrm{L}^2(\mathbb{R}^d)$ is $\mathcal{C}^1$ at 0 and the derivative, denoted T', is called shape derivative of T. In addition, for $\boldsymbol{\theta} \in \Theta_{\mathrm{ad}}$ and assuming that T_2 belongs to $\mathrm{H}^2(\Omega_2)$, the shape derivative $\mathsf{T}' \in \mathcal{H}_0(\Omega_1, \Omega_2)$ is characterized by,*

$$\left\{\begin{aligned} &\textit{Find } \mathsf{T}' \in \mathcal{H}_0(\Omega_1, \Omega_2), \ \textit{such that } \forall \phi \in \mathcal{H}_0(\Omega_1, \Omega_2), \\ &\sum_{i=1}^{2} \int_{\Omega_i} \kappa_i \nabla \mathsf{T}'_i \cdot \nabla \phi_i \,\mathrm{d}x + \int_{\Omega_1} \nabla \mathsf{T}'_1 \cdot \boldsymbol{u}\phi_1 \,\mathrm{d}x + \int_{\Gamma_{\mathrm{R}}} \alpha \mathsf{T}'_2 \phi_2 \,\mathrm{d}s \\ &\qquad + \int_{\Gamma} \kappa_{\mathrm{m}} \left(\epsilon \nabla_\tau \langle \mathsf{T}' \rangle \cdot \nabla_\tau \langle \phi \rangle + H[\mathsf{T}'] \langle \phi \rangle + \frac{1}{\epsilon}[\mathsf{T}'][\phi] \right) \mathrm{d}s \\ &= \int_{\Omega_2} \kappa_2 \left((\nabla\boldsymbol{\theta} + \nabla\boldsymbol{\theta}^t \operatorname{div}(\boldsymbol{\theta})\mathrm{I})\nabla \mathsf{T}_2 - \nabla(\boldsymbol{\theta} \cdot \nabla \mathsf{T}_2) \right) \cdot \nabla \phi_2 \,\mathrm{d}x \\ &\qquad - \int_{\Gamma_{\mathrm{R}}} \alpha \left(\operatorname{div}_\tau(\boldsymbol{\theta})(\mathsf{T}_2 - \mathsf{T}_{\mathrm{ext}}) + \boldsymbol{\theta} \cdot \nabla(\mathsf{T}_2 - \mathsf{T}_{\mathrm{ext}}) \right) \phi_2 \,\mathrm{d}s. \end{aligned}\right. \tag{2.4}$$

Furthermore, its strong form is given by,

$$
\begin{cases}
-\kappa_1 \Delta \mathsf{T}_1' + \nabla \mathsf{T}_1' \cdot \boldsymbol{u} &= 0 \text{ in } \Omega_1, \\
-\kappa_2 \Delta \mathsf{T}_2' &= 0 \text{ in } \Omega_2, \\
\mathsf{T}_1' &= 0 \text{ on } \Gamma_{\mathrm{D}}, \\
\kappa_1 \dfrac{\partial \mathsf{T}_1'}{\partial \boldsymbol{n}} &= 0 \text{ on } \Gamma_{\mathrm{N}}, \\
\kappa_2 \dfrac{\partial \mathsf{T}_2'}{\partial \boldsymbol{n}} &= 0 \text{ on } \Gamma_{\mathrm{e}}, \\
\kappa_2 \dfrac{\partial \mathsf{T}_2'}{\partial \boldsymbol{n}} + \alpha \mathsf{T}_2' &= \operatorname{div}_\tau((\boldsymbol{\theta}\cdot\boldsymbol{n})\kappa_2 \nabla_\tau \mathsf{T}_2) - \alpha(\boldsymbol{\theta}\cdot\boldsymbol{n}) \left(\dfrac{\partial}{\partial \boldsymbol{n}}(\mathsf{T}_2 - \mathsf{T}_{\mathrm{ext}}) + H(\mathsf{T}_2 - \mathsf{T}_{\mathrm{ext}}) \right) \text{ on } \Gamma_{\mathrm{R}}, \\
\left\langle \kappa \dfrac{\partial \mathsf{T}'}{\partial \boldsymbol{n}} \right\rangle &= -\dfrac{\kappa_{\mathrm{m}}}{\epsilon}[\mathsf{T}'] \text{ on } \Gamma, \\
\left[\kappa \dfrac{\partial \mathsf{T}'}{\partial \boldsymbol{n}} \right] &= \epsilon \kappa_{\mathrm{m}} \Delta_\tau \langle \mathsf{T}' \rangle - \kappa_{\mathrm{m}} H [\mathsf{T}'] \text{ on } \Gamma, \\
\kappa_i \dfrac{\partial \mathsf{T}_i'}{\partial \boldsymbol{n}} &= 0 \text{ on } \partial\Gamma, i = 1, 2.
\end{cases}
\tag{2.5}
$$

Finally we can state the result of shape differentiability concerning the objective functional.

Proposition 2.5 (Shape derivative of the functional)**.** *If* $\mathsf{T}_{\mathrm{ext}} \in \mathrm{H}^2(\mathbb{R}^d)$*, then the heat insulation functional* J *is shape differentiable in the direction* $\boldsymbol{\theta} \in \Theta_{\mathrm{ad}}$ *is given by*

$$
\begin{aligned}
J'(\Omega_2)(\boldsymbol{\theta}) = \int_{\Gamma_{\mathrm{R}}} & \operatorname{div}_\tau(\boldsymbol{\theta}) \left(\alpha^2 (\mathsf{T}_2 - \mathsf{T}_{\mathrm{ext}})^2 - \alpha (\mathsf{T}_2 - \mathsf{T}_{\mathrm{ext}}) \mathsf{R}_2 \right) \mathrm{d}s \\
& - \int_{\Gamma_{\mathrm{R}}} \left(2\alpha^2 (\mathsf{T}_2 - \mathsf{T}_{\mathrm{ext}}) + \alpha \mathsf{R}_2 \right) (\nabla \mathsf{T}_{\mathrm{ext}} \cdot \boldsymbol{\theta}) \, \mathrm{d}s \\
& \qquad + \int_{\Omega_2} \kappa_2 \left((\nabla \boldsymbol{\theta} + \nabla \boldsymbol{\theta}^t - \operatorname{div}(\boldsymbol{\theta}) \mathrm{I} \right) \nabla \mathsf{T}_2 \cdot \nabla \mathsf{R}_2 \, \mathrm{d}x,
\end{aligned}
\tag{2.6}
$$

where $\mathsf{T} = (\mathsf{T}_1, \mathsf{T}_2) \in \mathcal{H}_{\mathsf{T}_{\mathrm{D}}}(\Omega_1, \Omega_2)$ *is the solution of the convection-diffusion equation* (1.4) *and* $\mathsf{R} = (\mathsf{R}_1, \mathsf{R}_2) \in \mathcal{H}_0(\Omega_1, \Omega_2)$ *is the solution of the following adjoint equation*

$$
\begin{cases}
-\operatorname{div}(\kappa_1 \nabla \mathsf{R}_1 + \mathsf{R}_1 \boldsymbol{u}) &= 0 & \text{in } \Omega_1, \\
-\operatorname{div}(\kappa_2 \nabla \mathsf{R}_2) &= 0 & \text{in } \Omega_2, \\
\mathsf{R}_1 &= 0 & \text{on } \Gamma_{\mathrm{D}}, \\
\kappa_1 \dfrac{\partial \mathsf{R}_1}{\partial \boldsymbol{n}} + \mathsf{R}_1 \boldsymbol{u} \cdot \boldsymbol{n} &= 0 & \text{on } \Gamma_{\mathrm{N}}, \\
\kappa_2 \dfrac{\partial \mathsf{R}_2}{\partial \boldsymbol{n}} &= 0 & \text{on } \Gamma_{\mathrm{e}}, \\
\kappa_2 \dfrac{\partial \mathsf{R}_2}{\partial \boldsymbol{n}} + \alpha \mathsf{R}_2 &= 2\alpha^2 (\mathsf{T}_2 - \mathsf{T}_{\mathrm{ext}}) & \text{on } \Gamma_{\mathrm{R}}, \\
\left\langle \kappa \dfrac{\partial \mathsf{R}}{\partial \boldsymbol{n}} \right\rangle &= -\kappa_{\mathrm{m}} \epsilon^{-1} [\mathsf{R}] - \kappa_{\mathrm{m}} H \langle \mathsf{R} \rangle & \text{on } \Gamma, \\
\left[\kappa \dfrac{\partial \mathsf{R}}{\partial \boldsymbol{n}} \right] &= \epsilon \operatorname{div}_\tau(\kappa_{\mathrm{m}} \nabla_\tau \langle \mathsf{R} \rangle) & \text{on } \Gamma, \\
\kappa_i \dfrac{\partial \mathsf{R}_i}{\partial \boldsymbol{n}} &= 0 & \text{on } \partial\Gamma, i = 1, 2.
\end{cases}
\tag{2.7}
$$

If furthermore $\mathsf{T}_2, \mathsf{R}_2 \in \mathrm{H}^2(\Omega_2)$, then the shape derivative can be expressed on its surface form as

$$J'(\Omega_2)(\boldsymbol{\theta}) = \int_{\Gamma_\mathrm{R}} f(\mathsf{T}_2, \mathsf{R}_2)(\boldsymbol{\theta} \cdot \boldsymbol{n}) \, \mathrm{d}s, \tag{2.8}$$

with

$$f(\mathsf{T}_2, \mathsf{R}_2) = \alpha^2 (\mathsf{T}_2 - \mathsf{T}_\mathrm{ext})^2 \left(H - \frac{4\alpha}{\kappa_2} \right) + \alpha (\mathsf{T}_2 - \mathsf{T}_\mathrm{ext}) \mathsf{R}_2 \left(\frac{2\alpha}{\kappa_2} - H \right) - \kappa_2 \nabla \mathsf{T}_2 \cdot \nabla \mathsf{R}_2 + \frac{\partial \mathsf{T}_\mathrm{ext}}{\partial \boldsymbol{n}} \left(\alpha \mathsf{R}_2 - 2\alpha^2 (\mathsf{T}_2 - \mathsf{T}_\mathrm{ext}) \right).$$

Remark 2.6. *The adjoint equation* (2.7) *is well-posed by virtue of the Lax-Milgram theorem. The proof is similar to the proof of Theorem 2.2: the coercivity of the associated bilinear form is analogous to the coercivity of $a(\cdot, \cdot)$. It just remains to show the continuity of the associated linear form, which is direct.*

2.3 Shape sensitivity analysis with random exterior temperature

Notice that previously, we have assumed to know the exterior parameter T_ext precisely. We want now to consider the more realistic case of an imprecise knowledge of this parameter and we then aim to take into account uncertainties about this data T_ext. To do that we will assume that we have information about the uncertainties.

Let $(\Xi, \mathcal{A}, \mathbb{P})$ be a complete probability space. We consider the case where the exterior temperature $\mathsf{T}_\mathrm{ext}(\cdot, \cdot)$ is given as a random process in the Bochner space $\mathrm{L}^2(\Xi, \mathrm{H}^{1/2}(\Gamma_\mathrm{R}))$. For any $\omega \in \Xi$, the solution $(\mathsf{T}_1, \mathsf{T}_2)$ of the convection-diffusion problem (1.4) depends on $\mathsf{T}_\mathrm{ext}(\cdot, \omega)$ and then becomes a random process defined as the unique solution in $\mathcal{H}_{\mathsf{T}_\mathrm{D}}(\Omega_1, \Omega_2)$ to the following system

$$\left\{ \begin{array}{rcll} -\operatorname{div}(\kappa_1 \nabla \mathsf{T}_1(\cdot, \omega)) + \boldsymbol{u} \cdot \nabla \mathsf{T}_1(\cdot, \omega) &=& 0 & \text{in } \Omega_1, \\ -\operatorname{div}(\kappa_2 \nabla \mathsf{T}_2(\cdot, \omega)) &=& 0 & \text{in } \Omega_2, \\ \mathsf{T}_1(\cdot, \omega) &=& \mathsf{T}_\mathrm{D} & \text{on } \Gamma_\mathrm{D}, \\ \kappa_1 \dfrac{\partial \mathsf{T}_1(\cdot, \omega)}{\partial \boldsymbol{n}} &=& 0 & \text{on } \Gamma_\mathrm{N}, \\ \kappa_2 \dfrac{\partial \mathsf{T}_2(\cdot, \omega)}{\partial \boldsymbol{n}} &=& 0 & \text{on } \Gamma_\mathrm{e}, \\ \kappa_2 \dfrac{\partial \mathsf{T}_2(\cdot, \omega)}{\partial \boldsymbol{n}} + \alpha \mathsf{T}_2(\cdot, \omega) &=& \alpha \mathsf{T}_\mathrm{ext}(\cdot, \omega) & \text{on } \Gamma_\mathrm{R}, \\ \left\langle \kappa \dfrac{\partial \mathsf{T}(\cdot, \omega)}{\partial \boldsymbol{n}} \right\rangle &=& -\kappa_\mathrm{m} \epsilon^{-1} \left[\mathsf{T}(\cdot, \omega) \right] & \text{on } \Gamma, \\ \left[\kappa \dfrac{\partial \mathsf{T}(\cdot, \omega)}{\partial \boldsymbol{n}} \right] &=& \epsilon \operatorname{div}_\tau (\kappa_\mathrm{m} \nabla_\tau \langle \mathsf{T}(\cdot, \omega) \rangle) - \kappa_\mathrm{m} H [\mathsf{T}(\cdot, \omega)] & \text{on } \Gamma, \\ \kappa_i \dfrac{\partial \mathsf{T}_i(\cdot, \omega)}{\partial \boldsymbol{n}} &=& 0 & \text{on } \partial\Gamma, i = 1, 2. \end{array} \right. \tag{2.9}$$

Since in this part we are interested in the study of the effect of a random exterior temperature, we will make explicit the dependence on T_ext by means of the notation:

$$J(\Omega_2, \mathsf{T}_\mathrm{ext}) = \int_{\Gamma_\mathrm{R}} \alpha^2 (\mathsf{T}_2 - \mathsf{T}_\mathrm{ext})^2 \, \mathrm{d}s.$$

The objective is now to minimize $\mathbb{E}[J(\Omega_2, \mathsf{T}_\mathrm{ext}(x, \cdot))]$ the expectation of the objective functional J. The functional of interest J is quadratic in a temperature that depends linearly on the random parameter. This situation fits in the context of the work of Dambrine *et al.* [15]. Note that considering higher order moments can been done by a mere adaptation of the methods (see [16]). The gradient of $\mathbb{E}[J(\Omega_2, \mathsf{T}_{\rm ext}(x,\cdot))]$ can be computed thanks to the two-point correlation of the random input $\mathsf{T}_{\rm ext}$. In order to avoid the needed introduction of tensor calculus for the general case (see [15] for the quadratic case and [13] for the general polynomial case), we restrict ourselves in this work to the particular case where $\mathsf{T}_{\rm ext}$ is a finite sum

$$\mathsf{T}_{\rm ext}(x,\omega) = \mathsf{T}^0_{\rm ext}(x) + \sum_{k=1}^{m} \xi_k(\omega)\mathsf{T}^k_{\rm ext}(x), \quad x \in \Omega,\ \omega \in \Xi, \tag{2.10}$$

where, for each $k = 1, \dots, m$, $\mathsf{T}^k \in \mathcal{H}_0(\Omega_1, \Omega_2)$ solves Equation (1.4) with exterior temperature $\mathsf{T}^k_{\rm ext}$ on $\Gamma_{\rm R}$ and 0 as Dirichlet boundary condition on $\Gamma_{\rm D}$, and where $\mathsf{T}^0 \in \mathcal{H}_{\mathsf{T}_{\rm D}}(\Omega_1, \Omega_2)$ solves Equation (1.4) with exterior temperature $\mathsf{T}^0_{\rm ext}$ on $\Gamma_{\rm R}$ and $\mathsf{T}_{\rm D}$ as Dirichlet boundary condition on $\Gamma_{\rm D}$. The random variables ξ_k are assumed independent following centered Gaussian distributions with variance σ_k^2. This case can be easily treated, but remains representative of the general situation when the number m of terms goes to ∞. The expression (2.10) is a so-called truncated Karhunen-Loève decomposition (see, e.g., [15, 3] about this expansion and its use in reliability-based optimization).

Theorem 2.7 (Shape derivative of the expectation of J)**.** *Let us consider an uncertain exterior temperature expanded as in* (2.10)*. Let us also assume that the random variables ξ_k have zero expected value and are independent. Then the expectation of J can be computed as*

$$\mathbb{E}[J(\Omega_2, \mathsf{T}_{\rm ext})] = J(\Omega_2, \mathsf{T}^0_{\rm ext}) + \sum_{k=1}^{m} \sigma_k^2 J(\Omega_2, \mathsf{T}^k_{\rm ext}) \tag{2.11}$$

and, under regularity assumptions similar to those of Proposition 2.5, its shape derivative is then given by

$$\left(\mathbb{E}[J]\right)'(\Omega_2, \mathsf{T}_{\rm ext})(\boldsymbol{\theta}) = J'(\Omega_2, \mathsf{T}^0_{\rm ext})(\boldsymbol{\theta}) + \sum_{k=1}^{m} \sigma_k^2 J'(\Omega_2, \mathsf{T}^k_{\rm ext})(\boldsymbol{\theta}). \tag{2.12}$$

3 Proofs

3.1 Proof of the well-posedness theorem 2.2

Proof of Theorem 2.2. We follow the usual strategy: lift the boundary condition and apply Lax-Milgram theorem in the space $\mathcal{H}_0(\Omega_1, \Omega_2)$. The crucial point is to prove that a is coercive. The presence of an interface condition on Γ makes it not completely customary. We therefore demonstrate this point.

Let $\mathsf{S} \in \mathcal{H}_0(\Omega_1, \Omega_2)$. We split $a(\mathsf{S},\mathsf{S})$ into $a_1(\mathsf{S},\mathsf{S}) + a_2(\mathsf{S},\mathsf{S}) + a_3(\mathsf{S},\mathsf{S})$ where

$$\begin{aligned}
a_1(\mathsf{S},\mathsf{S}) &:= \sum_{i=1}^{2} \int_{\Omega_i} \kappa_i |\nabla \mathsf{S}_i|^2 \,\mathrm{d}x + \int_{\Gamma} \kappa_{\rm m} \left(\epsilon |\nabla_\tau \langle \mathsf{S} \rangle|^2 + \frac{1}{\epsilon}[\mathsf{S}]^2 \right) \mathrm{d}s + \int_{\Gamma_{\rm R}} \alpha \mathsf{S}_2^2 \,\mathrm{d}s,\\
a_2(\mathsf{S},\mathsf{S}) &:= \int_{\Omega_1} \mathsf{S}_1 (\boldsymbol{u} \cdot \nabla \mathsf{S}_1) \,\mathrm{d}x,\\
a_3(\mathsf{S},\mathsf{S}) &:= \int_{\Gamma} \kappa_{\rm m} H [\mathsf{S}] \langle \mathsf{S} \rangle \,\mathrm{d}s.
\end{aligned}$$

The bilinear form a_1 clearly is coercive,

$$a_1(\mathsf{S},\mathsf{S}) \geq \sum_{i=1}^{2} \|\kappa_i^{1/2} \nabla \mathsf{S}_i\|^2_{\mathrm{L}^2(\Omega_i)^d} + \epsilon\kappa_{\rm m} \|\nabla_\tau \langle \mathsf{S} \rangle\|^2_{\mathrm{L}^2(\Gamma)} + \frac{\kappa_{\rm m}}{\epsilon} \|[\mathsf{S}]\|^2_{\mathrm{L}^2(\Gamma)}.$$

Concerning a_2, we get after integration by parts

$$a_2(\mathsf{S},\mathsf{S}) = \int_{\Omega_1} \boldsymbol{u}\cdot\nabla\left(\frac{\mathsf{S}^2}{2}\right)\,\mathrm{d}x = \frac{1}{2}\int_{\Gamma_\mathrm{N}} \mathsf{S}_2^2\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}s.$$

We have used the boundary conditions $\boldsymbol{u}=0$ on Γ and $\mathsf{S}_1=0$ on Γ_D and the incompressiblility of the fluid $\mathrm{div}(\boldsymbol{u})=0$ in Ω_1. Now since the output normal velocity $\boldsymbol{u}\cdot\boldsymbol{n}$ is nonnegative on Γ_N by the exit condition (2.2), we get $a_2(\mathsf{S},\mathsf{S})\geq 0$. The difficulty lies in the product $[\mathsf{S}]\langle\mathsf{S}\rangle$ that has no definite sign. Using successively Cauchy-Schwarz then Young inequalities, one gets

$$\begin{aligned}
\left|\int_\Gamma H[\mathsf{S}]\,\langle\mathsf{S}\rangle\,\mathrm{d}s\right| &= \frac{1}{2}\left|\int_\Gamma [\mathsf{S}]\ \left(H\ \mathsf{S}_1\ +\ H\ \mathsf{S}_2\right)\,\mathrm{d}s\right| \\
&\leq \frac{1}{2}\|[\mathsf{S}]\|^2_{\mathrm{L}^2(\Gamma)}\left(\|H\mathsf{S}_1\|^2_{\mathrm{L}^2(\Gamma)}+\|H\mathsf{S}_2\|^2_{\mathrm{L}^2(\Gamma)}\right) \\
&\leq \frac{1}{2}\left(\frac{1}{2\epsilon}\|[\mathsf{S}]\|^2_{\mathrm{L}^2(\Gamma)}+\frac{\epsilon}{2}\|H\mathsf{S}_1\|^2_{\mathrm{L}^2(\Gamma)}+\frac{1}{2\epsilon}\|[\mathsf{S}]\|^2_{\mathrm{L}^2(\Gamma)}+\frac{\epsilon}{2}\|H\mathsf{S}_2\|^2_{\mathrm{L}^2(\Gamma)}\right) \\
&\leq \frac{1}{2\epsilon}\|[\mathsf{S}]\|^2_{\mathrm{L}^2(\Gamma)}+\frac{\epsilon\|H\|^2_\infty}{4}\left(\|\mathsf{S}_1\|^2_{\mathrm{L}^2(\Gamma)}+\|\mathsf{S}_2\|^2_{\mathrm{L}^2(\Gamma)}\right).
\end{aligned}$$

Since $\mathsf{S}_1=0$ on Γ_D, one infers from the trace theorem and Poincaré inequality for S_1, the existence of a positive constant $C>0$, such that, $\|\mathsf{S}_1\|^2_{\mathrm{L}^2(\Gamma)}\leq C\|\nabla\mathsf{S}_1\|^2_{\mathrm{L}^2(\Omega_1)^d}$. This is not the case for S_2. Using the definition of the jump, one gets

$$\|\mathsf{S}_2\|_{\mathrm{L}^2(\Gamma)}\leq\|\mathsf{S}_1\|_{\mathrm{L}^2(\Gamma)}+\|[\mathsf{S}]\|_{\mathrm{L}^2(\Gamma)}\ \text{then}\ \|\mathsf{S}_2\|^2_{\mathrm{L}^2(\Gamma)}\leq 2\left(\|\mathsf{S}_1\|^2_{\mathrm{L}^2(\Gamma)}+\|[\mathsf{S}]\|^2_{\mathrm{L}^2(\Gamma)}\right)$$

by the triangle inequality. Finally, we have obtained the bound

$$\left|\int_\Gamma H[\mathsf{S}]\,\langle\mathsf{S}\rangle\,\mathrm{d}s\right|\leq\frac{1}{2}\left(\frac{1}{\epsilon}+\epsilon\|H\|^2_\infty\right)\|[\mathsf{S}]\|^2_{\mathrm{L}^2(\Gamma)}+\frac{3\epsilon C\|H\|^2_\infty}{4}\,\|\nabla\mathsf{S}_1\|^2_{\mathrm{L}^2(\Omega_1)^d}.$$

Therefore,

$$\begin{aligned}
a_1(\mathsf{S},\mathsf{S})+a_3(\mathsf{S},\mathsf{S})\geq\sum_{i=1}^2\|\kappa_i^{1/2}\nabla\mathsf{S}_i\|^2_{\mathrm{L}^2(\Omega_i)^d}-\ \epsilon\ \frac{3C\kappa_\mathrm{m}\|H\|^2_\infty}{4}\ \|\nabla\mathsf{S}_1\|^2_{\mathrm{L}^2(\Omega_1)^d}+\epsilon\kappa_\mathrm{m}\|\nabla_\tau\langle\mathsf{S}\rangle\|^2_{\mathrm{L}^2(\Gamma)^d} \\
+\frac{\kappa_\mathrm{m}}{2}\left(\frac{1}{\epsilon}-\epsilon\|H\|^2_\infty\right)\|[\mathsf{S}]\|^2_{\mathrm{L}^2(\Gamma)}.
\end{aligned}$$

We impose $\epsilon<\|H\|^{-1}_\infty$ so that the last term is nonnegative. The second term is absorbed by the correspond term in a_1 if we impose that

$$\kappa_1-\epsilon\ \frac{3C\kappa_\mathrm{m}\|H\|^2_\infty}{4}\geq\frac{1}{2}\kappa_1\Leftrightarrow\epsilon\leq\frac{2\kappa_1}{3C\kappa_\mathrm{m}\|H\|^2_\infty}.$$

In conclusion, a is coercive if

$$\epsilon<\epsilon_0:=\min\left(\frac{1}{\|H\|_\infty},\frac{2\kappa_1}{3C\kappa_\mathrm{m}\|H\|^2_\infty}\right),$$

which concludes the proof.

□

3.2 Shape sensitivity analysis

Before proving the main result of this part (Proposition 2.5), we need some auxiliary results, such as the existence of the derivative. As it is classical in shape optimization, the first step is to show the existence of the material derivative and then compute it (see, e.g., [23]). For the sake of simplicity, we assume without loss of generality that $\mathsf{T}_\mathrm{D} = 0$.

We recall that, for $\boldsymbol{\theta} \in \Theta_\mathrm{ad}$, $\mathsf{T}_{\boldsymbol{\theta}} \in \mathcal{H}_{\mathsf{T}_\mathrm{D}}(\Omega_1, \Omega_2^{\boldsymbol{\theta}})$ is the solution of the convection-diffusion Problem (1.4) defined on $\Omega_1 \cup \Omega_2^{\boldsymbol{\theta}}$ instead of $\Omega_1 \cup \Omega_2$. Throughout this section, $\boldsymbol{u} \in \mathcal{V}_{\boldsymbol{u}_\mathrm{D}}(\Omega_1)$ is the velocity solution of the incompressible Navier-Stokes equations (1.3), which does not depend on $\boldsymbol{\theta}$, since Ω_1 is fixed.

Proposition 3.1 (Existence and characterization of the material derivative of T)**.** *For all $\boldsymbol{\theta} \in \Theta_\mathrm{ad}$, we define $\overline{\mathsf{T}}_{\boldsymbol{\theta}} := \mathsf{T}_{\boldsymbol{\theta}} \circ (\boldsymbol{I} + \boldsymbol{\theta}) \in \mathcal{H}_{\mathsf{T}_\mathrm{D}}(\Omega_1, \Omega_2)$. If $\mathsf{T}_\mathrm{ext} \in \mathrm{H}^2(\mathbb{R}^d)$, then*

$$\boldsymbol{\theta} \in \Theta_\mathrm{ad} \mapsto \overline{\mathsf{T}}_{\boldsymbol{\theta}} \in \mathcal{H}_{\mathsf{T}_\mathrm{D}}(\Omega_1, \Omega_2)$$

is differentiable in a neighborhood of 0. Furthermore, its derivative at 0, in the direction $\boldsymbol{\theta}$, is called the material derivative of $\mathsf{T} \in \mathcal{H}_{\mathsf{T}_\mathrm{D}}(\Omega_1, \Omega_2)$, is denoted by $\dot{\mathsf{T}} \in \mathcal{H}_0(\Omega_1, \Omega_2)$, and is the solution of the following variational problem

$$\left\{\begin{array}{l}\text{Find } \dot{\mathsf{T}} \in \mathcal{H}_0(\Omega_1, \Omega_2) \text{ such that, for all } \phi \in \mathcal{H}_0(\Omega_1, \Omega_2),\\ \displaystyle\sum_{i=1}^{2} \int_{\Omega_i} \kappa_i \nabla \dot{\mathsf{T}}_i \cdot \nabla \phi_i \,\mathrm{d}x + \int_{\Omega_1} \nabla \dot{\mathsf{T}}_1 \cdot \boldsymbol{u} \phi_1 \,\mathrm{d}x + \int_{\Gamma_\mathrm{R}} \alpha \dot{\mathsf{T}}_2 \phi_2 \,\mathrm{d}s\\ \quad \displaystyle + \int_{\Gamma} \kappa_\mathrm{m} \left(\epsilon \nabla_\tau \left\langle \dot{\mathsf{T}} \right\rangle \cdot \nabla_\tau \langle \phi \rangle + H[\dot{\mathsf{T}}] \langle \phi \rangle + \frac{1}{\epsilon}[\dot{\mathsf{T}}][\phi] \right) \mathrm{d}s\\ \displaystyle = \int_{\Omega_2} \kappa_2 \left(\nabla \boldsymbol{\theta} + \nabla \boldsymbol{\theta}^t - \mathrm{div}(\boldsymbol{\theta}) \mathrm{I} \right) \nabla \mathsf{T}_2 \cdot \nabla \phi_2 \,\mathrm{d}x - \int_{\Gamma_\mathrm{R}} \alpha \left(\mathrm{div}_\tau(\boldsymbol{\theta})(\mathsf{T}_2 - \mathsf{T}_\mathrm{ext}) - \nabla \mathsf{T}_\mathrm{ext} \cdot \boldsymbol{\theta} \right) \phi_2 \,\mathrm{d}s.\end{array}\right. \tag{3.1}$$

Proof of Proposition 3.1. We proceed as described in [23]. Let $\phi \in \mathcal{H}_0(\Omega_1, \Omega_2)$ and let $\boldsymbol{\theta} \in \Theta_\mathrm{ad}$. We define $\phi_{\boldsymbol{\theta}} := \phi \circ (\boldsymbol{I} + \boldsymbol{\theta})^{-1} \in \mathcal{H}_0(\Omega_1, \Omega_2^{\boldsymbol{\theta}})$ and we have

$$\begin{aligned}&\int_{\Omega_2^{\boldsymbol{\theta}}} \kappa_2 \nabla \mathsf{T}_{\boldsymbol{\theta},2} \cdot \nabla \phi_{\boldsymbol{\theta},2} \,\mathrm{d}x + \int_{\Omega_1} \left(\kappa_1 \nabla \mathsf{T}_{\boldsymbol{\theta},1} \cdot \nabla \phi_1 + \nabla \mathsf{T}_{\boldsymbol{\theta},1} \cdot \boldsymbol{u} \phi_1 \right) \mathrm{d}x + \int_{\Gamma_\mathrm{R}} \alpha \mathsf{T}_{\boldsymbol{\theta},2} \phi_{\boldsymbol{\theta},2} \,\mathrm{d}s\\ &\qquad + \int_{\Gamma^{\boldsymbol{\theta}}} \kappa_\mathrm{m} \left(\epsilon \nabla_{\tau_{\boldsymbol{\theta}}} \langle \mathsf{T}_{\boldsymbol{\theta}} \rangle \cdot \nabla_{\tau_{\boldsymbol{\theta}}} \langle \phi_{\boldsymbol{\theta}} \rangle + H_{\boldsymbol{\theta}}[\mathsf{T}_{\boldsymbol{\theta}}] \langle \phi_{\boldsymbol{\theta}} \rangle + \epsilon^{-1} [\mathsf{T}_{\boldsymbol{\theta}}][\phi_{\boldsymbol{\theta}}] \right) \mathrm{d}s = \int_{\Gamma_\mathrm{R}^{\boldsymbol{\theta}}} \alpha \mathsf{T}_\mathrm{ext} \phi_{\boldsymbol{\theta},2} \,\mathrm{d}s,\end{aligned}$$

where we have used that $\Omega_1^{\boldsymbol{\theta}} = \Omega_1$ and $\boldsymbol{u} = \boldsymbol{u} \circ (\mathrm{I} + \boldsymbol{\theta})$, since $\boldsymbol{\theta} = 0$ in $\overline{\Omega}_1$. Changing variables, we get

$$\begin{aligned}&\int_{\Omega_2} \kappa_2 A(\boldsymbol{\theta}) \nabla \overline{\mathsf{T}}_{\boldsymbol{\theta},2} \cdot \nabla \phi_2 \,\mathrm{d}x + \int_{\Omega_1} \left(\kappa_1 \nabla \overline{\mathsf{T}}_{\boldsymbol{\theta},1} \cdot \nabla \phi_1 + \nabla \overline{\mathsf{T}}_{\boldsymbol{\theta},1} \cdot \boldsymbol{u} \phi_1 \right) \mathrm{d}x + \int_{\Gamma_\mathrm{R}} \alpha B(\boldsymbol{\theta}) \overline{\mathsf{T}}_{\boldsymbol{\theta},2} \phi_2 \,\mathrm{d}s\\ &+ \int_{\Gamma} \kappa_\mathrm{m} \left(\epsilon \nabla_\tau \left\langle \overline{\mathsf{T}}_{\boldsymbol{\theta}} \right\rangle \cdot \nabla_\tau \langle \phi \rangle + H[\overline{\mathsf{T}}_{\boldsymbol{\theta}}] \langle \phi \rangle + \epsilon^{-1} [\overline{\mathsf{T}}_{\boldsymbol{\theta}}][\phi] \right) \mathrm{d}s = \int_{\Gamma_\mathrm{R}} \alpha B(\boldsymbol{\theta}) \mathsf{T}_\mathrm{ext} \circ (\boldsymbol{I} + \boldsymbol{\theta}) \phi_2 \,\mathrm{d}s,\end{aligned} \tag{3.2}$$

where

$$A(\boldsymbol{\theta}) := |\det(\mathrm{I} + \nabla \boldsymbol{\theta})| (\mathrm{I} + \nabla \boldsymbol{\theta})^{-1} (\mathrm{I} + \nabla \boldsymbol{\theta})^{-t}, B(\boldsymbol{\theta}) := |\det(\mathrm{I} + \nabla \boldsymbol{\theta})| |(\mathrm{I} + \nabla \boldsymbol{\theta})^{-t} \boldsymbol{n}|_{\mathbb{R}^d}, \tag{3.3}$$

and $|\cdot|_{\mathbb{R}^d}$ is the usual Eucledian norm in $\mathbb{R}^d$. Then, we introduce $\mathcal{F} : \Theta_\mathrm{ad} \times \mathcal{H}_0(\Omega_1, \Omega_2) \mapsto (\mathcal{H}_0(\Omega_1, \Omega_2))'$, defined for all $S \in \mathcal{H}_0(\Omega_1, \Omega_2)$ by,

$$\langle \mathcal{F}(\boldsymbol{\theta},\overline{T}),S\rangle := \int_{\Omega_2} \kappa_2 A(\boldsymbol{\theta})\nabla\overline{T}_2\cdot\nabla S_2 \,\mathrm{d}x + \int_{\Omega_1} \left(\kappa_1\nabla\overline{T}_1\cdot\nabla S_1 + \nabla\overline{T}_1\cdot\boldsymbol{u}S_1\right)\mathrm{d}x + \int_{\Gamma_\mathrm{R}} \alpha B(\boldsymbol{\theta})\overline{T}_2 S_2 \,\mathrm{d}s$$
$$+\int_\Gamma \kappa_\mathrm{m}\left(\epsilon\nabla_\tau\left\langle\overline{T}\right\rangle\cdot\nabla_\tau\langle S\rangle + H[\overline{T}]\langle S\rangle + \frac{1}{\epsilon}[\overline{T}][S]\right)\mathrm{d}s - \int_{\Gamma_\mathrm{R}} \alpha B(\boldsymbol{\theta})\mathsf{T}_\mathrm{ext}\circ(\boldsymbol{I}+\boldsymbol{\theta})S_2\,\mathrm{d}s.$$

By construction $\mathcal{F}(0,\mathsf{T})=0$ where $\mathsf{T}\in\mathcal{H}_0(\Omega_1,\Omega_2)$ is the solution of the approximate convection-diffusion equation (1.4). Similarly to [23, Theorem 5.5.1], we show that $\mathcal{F}$ is $\mathcal{C}^1$. Finally, we compute $D_{\overline{T}}\mathcal{F}(0,\mathsf{T})$ that for all $S,\hat{S}\in\mathcal{H}_0(\Omega_1,\Omega_2)$, which is given by

$$\langle D_{\overline{T}}\mathcal{F}(0,\mathsf{T})S,\hat{S}\rangle = \sum_{i=1}^{2}\int_{\Omega_i}\kappa_i\nabla S_i\cdot\nabla\hat{S}_i\,\mathrm{d}x + \int_{\Omega_1}\nabla S_1\cdot\boldsymbol{u}\hat{S}_1\,\mathrm{d}x$$
$$+\int_{\Gamma_\mathrm{R}}\alpha S_2\hat{S}_2\,\mathrm{d}s + \int_\Gamma \kappa_\mathrm{m}\left(\epsilon\nabla_\tau\langle S\rangle\cdot\nabla_\tau\langle\hat{S}\rangle + H[S][\hat{S}] + \frac{1}{\epsilon}[S][\hat{S}]\right)\mathrm{d}s.$$

Hence, thanks to the well-posedness Theorem 2.2 we deduce that $D_{\overline{T}}\mathcal{F}(0,\mathsf{T})$ is an isomorphism from $\mathcal{H}_0(\Omega_1,\Omega_2)$ into $(\mathcal{H}_0(\Omega_1,\Omega_2))'$.

By virtue of the implicit function theorem, there exists a $\mathcal{C}^1$ function $\boldsymbol{\theta}\in\Theta_\mathrm{ad}\mapsto\overline{T}(\boldsymbol{\theta})\in\mathcal{H}_0(\Omega_1,\Omega_2)$ in a neighborhood of 0 such that, $\mathcal{F}(\boldsymbol{\theta},\overline{T}(\boldsymbol{\theta}))=0$. By uniqueness of the solution $\mathsf{T}_{\boldsymbol{\theta}}\in\mathcal{H}_0(\Omega_1,\Omega_2^{\boldsymbol{\theta}})$ (Theorem 2.2) and from (3.2), we deduce $\overline{\mathsf{T}}_{\boldsymbol{\theta}}=\overline{T}(\boldsymbol{\theta})$, then, $\boldsymbol{\theta}\in\Theta_\mathrm{ad}\mapsto\overline{\mathsf{T}}_{\boldsymbol{\theta}}\in\mathcal{H}_0(\Omega_1,\Omega_2)$ is $\mathcal{C}^1$.

To prove that the material derivative $\dot{\mathsf{T}}\in\mathcal{H}_0(\Omega_1,\Omega_2)$ satisfies (3.1), we proceed as in [1, Proposition 6.30]. We first recall that

$$A'(0)(\boldsymbol{\theta}) = \mathrm{div}(\boldsymbol{\theta})\mathrm{I}-\nabla\boldsymbol{\theta}-(\nabla\boldsymbol{\theta})^t,\quad B'(0)(\boldsymbol{\theta})=\mathrm{div}_\tau(\boldsymbol{\theta})\quad\text{ and }\quad\overline{\mathsf{T}}'(0)(\boldsymbol{\theta})=\dot{\mathsf{T}}.$$

Then, differentiating (3.2) at $\boldsymbol{\theta}=0$, in the direction $\boldsymbol{\theta}$ and using the chain rule of those derivatives, we get (3.1). □

After showing the existence and computing the material derivative, we can do the same for the Eulerian derivative, whose proof uses the previous results and some integrations by parts.

Proof of Proposition 2.4. Let us introduce a linear continuous extension

$$E:\mathcal{H}_0(\Omega_1,\Omega_2)\mapsto\mathrm{H}^1(\mathbb{R}^d)\times\mathrm{H}^1(\mathbb{R}^d).$$

We define $\tilde{\mathsf{T}}_{\boldsymbol{\theta}} := E(\overline{\mathsf{T}}_{\boldsymbol{\theta}})\circ(\boldsymbol{I}+\boldsymbol{\theta})^{-1}$ and since $\boldsymbol{\theta}\in\Theta_\mathrm{ad}\mapsto\overline{\mathsf{T}}_{\boldsymbol{\theta}}\in\mathcal{H}_0(\Omega_1,\Omega_2)$ is differentiable in a neighborhood of 0, we obtain the existence of the shape derivative by using [23, Lemma 5.3.3].

Since $\mathsf{T}_2\in\mathrm{H}^2(\Omega_2)$, we have by definition of the Eulerian derivative that for any $\boldsymbol{\theta}\in\Theta_\mathrm{ad}$,

$$\mathsf{T}_1'=\dot{\mathsf{T}}_1\in\mathrm{H}^1(\Omega_1),\ \mathsf{T}_2'=\dot{\mathsf{T}}_2-\boldsymbol{\theta}\cdot\nabla\mathsf{T}_2\in\mathrm{H}^1(\Omega_2),\ \langle\mathsf{T}'\rangle=\left\langle\dot{\mathsf{T}}\right\rangle\in\mathrm{H}^1(\Gamma).$$

Using this in the problem solved by the material derivative (3.1), leads to $\mathsf{T}'\in\mathcal{H}_0(\Omega_1,\Omega_2)$ solves (2.4). In order to obtain the strong form of the shape derivative (2.5), we integrate by parts in formula (2.4), this is,

$$\int_{\Omega_2}\kappa_2\nabla\mathsf{T}_2'\cdot\nabla\phi_2\,\mathrm{d}x+\int_{\Omega_1}\left(\kappa_1\nabla\mathsf{T}_1'\cdot\nabla\phi_1+\phi_1\nabla\mathsf{T}_1'\cdot\boldsymbol{u}\right)\mathrm{d}x+\int_{\Gamma_\mathrm{R}}\alpha\mathsf{T}_2'\phi_2\,\mathrm{d}s$$
$$+\int_\Gamma\kappa_\mathrm{m}\left(\epsilon\nabla_\tau\langle\mathsf{T}'\rangle\cdot\nabla_\tau\langle\phi\rangle+H[\mathsf{T}']\langle\phi\rangle+\epsilon^{-1}[\mathsf{T}'][\phi]\right)\mathrm{d}s$$

$$
\begin{aligned}
&= \int_{\Omega_2} \kappa_2 \left((\nabla\boldsymbol{\theta} + \nabla\boldsymbol{\theta}^t)\nabla\mathsf{T}_2 - \operatorname{div}(\boldsymbol{\theta})\nabla\mathsf{T}_2 - \nabla(\boldsymbol{\theta}\cdot\nabla\mathsf{T}_2)\right)\cdot\nabla\phi_2 \,\mathrm{d}x \\
&\qquad - \int_{\Gamma_\mathrm{R}} \alpha\left(\operatorname{div}_\tau(\boldsymbol{\theta})(\mathsf{T}_2 - \mathsf{T}_\mathrm{ext}) + \boldsymbol{\theta}\cdot\nabla(\mathsf{T}_2 - \mathsf{T}_\mathrm{ext})\right)\phi_2 \,\mathrm{d}s \\
&= \int_{\Omega_2} \kappa_2 \left((\nabla\boldsymbol{\theta} - \operatorname{div}(\boldsymbol{\theta})\mathrm{I})\nabla\mathsf{T}_2 - (\nabla^2\mathsf{T}_2)\boldsymbol{\theta}\right)\cdot\nabla\phi_2 \,\mathrm{d}x \\
&\qquad - \int_{\Gamma_\mathrm{R}} \alpha\left(\operatorname{div}_\tau(\boldsymbol{\theta})(\mathsf{T}_2 - \mathsf{T}_\mathrm{ext}) + \boldsymbol{\theta}\cdot\nabla(\mathsf{T}_2 - \mathsf{T}_\mathrm{ext})\right)\phi_2 \,\mathrm{d}s \\
&= \int_{\Omega_2} \left(\kappa_2 \operatorname{div}((\boldsymbol{\theta}\cdot\nabla\phi_2)\nabla\mathsf{T}_2 - (\nabla\mathsf{T}_2\cdot\nabla\phi_2)\boldsymbol{\theta}) - (\boldsymbol{\theta}\cdot\nabla\phi_2)\kappa_2\Delta\mathsf{T}_2\right) \mathrm{d}x \\
&\qquad - \int_{\Gamma_\mathrm{R}} \alpha\left(\operatorname{div}_\tau(\boldsymbol{\theta})(\mathsf{T}_2 - \mathsf{T}_\mathrm{ext}) + \boldsymbol{\theta}\cdot\nabla(\mathsf{T}_2 - \mathsf{T}_\mathrm{ext})\right)\phi_2 \,\mathrm{d}s.
\end{aligned}
$$

Since $\mathsf{T} \in \mathcal{H}_{\mathsf{T}_\mathrm{D}}(\Omega_1, \Omega_2)$ is the solution of the convection-diffusion equation (1.4), $\kappa_2\Delta\mathsf{T}_2 = 0$ in Ω_2 with the boundary condition $\kappa_2 \frac{\partial \mathsf{T}_2}{\partial \boldsymbol{n}} = \alpha(\mathsf{T}_\mathrm{ext} - \mathsf{T}_2)$ on Γ_R, then

$$
\begin{aligned}
&\int_{\Omega_2} \kappa_2 \nabla\mathsf{T}_2'\cdot\nabla\phi_2 \,\mathrm{d}x + \int_{\Omega_1} \left(\kappa_1\nabla\mathsf{T}_1'\cdot\nabla\phi_1 + \phi_1\nabla\mathsf{T}_1'\cdot\boldsymbol{u}\right) \mathrm{d}x + \int_{\Gamma_\mathrm{R}} \alpha\mathsf{T}_2'\phi_2 \,\mathrm{d}s \\
&\qquad + \int_\Gamma \kappa_\mathrm{m}\left(\epsilon\nabla_\tau\langle\mathsf{T}'\rangle\cdot\langle\phi\rangle + H[\mathsf{T}']\langle\phi\rangle + \epsilon^{-1}[\mathsf{T}'][\phi]\right) \mathrm{d}s \\
&\qquad = \int_{\Omega_2} \kappa_2 \operatorname{div}((\boldsymbol{\theta}\cdot\nabla\phi_2)\nabla\mathsf{T}_2 - (\nabla\mathsf{T}_2\cdot\nabla\phi_2)\boldsymbol{\theta}) \,\mathrm{d}x \\
&\qquad\qquad - \int_{\Gamma_\mathrm{R}} \alpha\left(\operatorname{div}_\tau((\mathsf{T}_2 - \mathsf{T}_\mathrm{ext})\boldsymbol{\theta}) + (\boldsymbol{\theta}\cdot\boldsymbol{n})\frac{\partial}{\partial\boldsymbol{n}}(\mathsf{T}_2 - \mathsf{T}_\mathrm{ext})\right)\phi_2 \,\mathrm{d}s.
\end{aligned}
$$

By divergence theorem and the fact that $\boldsymbol{\theta} = 0$ on Γ,

$$
\begin{aligned}
&\int_{\Omega_2} \kappa_2 \nabla\mathsf{T}_2'\cdot\nabla\phi_2 \,\mathrm{d}x + \int_{\Omega_1} \left(\kappa_1\nabla\mathsf{T}_1'\cdot\nabla\phi_1 + \phi_1\nabla\mathsf{T}_1'\cdot\boldsymbol{u}\right) \mathrm{d}x + \int_{\Gamma_\mathrm{R}} \alpha\mathsf{T}_2'\phi_2 \,\mathrm{d}s \\
&\qquad + \int_\Gamma \kappa_\mathrm{m}\left(\epsilon\nabla_\tau\langle\mathsf{T}'\rangle\cdot\langle\phi\rangle + H[\mathsf{T}']\langle\phi\rangle + \epsilon^{-1}[\mathsf{T}'][\phi]\right) \mathrm{d}s \\
&= \int_{\Gamma_\mathrm{R}} \left((\boldsymbol{\theta}\cdot\nabla\phi_2)\kappa_2\frac{\partial\mathsf{T}_2}{\partial\boldsymbol{n}} - \kappa_2\nabla\mathsf{T}_2\cdot\nabla\phi_2 - \alpha\operatorname{div}_\tau((\mathsf{T}_2 - \mathsf{T}_\mathrm{ext})\boldsymbol{\theta})\phi_2 - \alpha(\boldsymbol{\theta}\cdot\boldsymbol{n})\phi_2\frac{\partial}{\partial\boldsymbol{n}}(\mathsf{T}_2 - \mathsf{T}_\mathrm{ext})\right) \mathrm{d}s.
\end{aligned}
$$

Decomposing the gradient as $\nabla\phi = \nabla_\tau\phi + \boldsymbol{n}\frac{\partial\phi}{\partial\boldsymbol{n}}$ and using again that T verifies the Robin boundary condition at Γ_R, we obtain that

$$
\begin{aligned}
&\int_{\Omega_2} \kappa_2 \nabla\mathsf{T}_2'\cdot\nabla\phi_2 \,\mathrm{d}x + \int_{\Omega_1} \left(\kappa_1\nabla\mathsf{T}_1'\cdot\nabla\phi_1 + \phi_1\nabla\mathsf{T}_1'\cdot\boldsymbol{u}\right) \mathrm{d}x + \int_{\Gamma_\mathrm{R}} \alpha\mathsf{T}_2'\phi_2 \,\mathrm{d}s \\
&\qquad + \int_\Gamma \kappa_\mathrm{m}\left(\epsilon\nabla_\tau\langle\mathsf{T}'\rangle\cdot\nabla_\tau\langle\phi\rangle + H[\mathsf{T}']\langle\phi\rangle + \epsilon^{-1}[\mathsf{T}'][\phi]\right) \mathrm{d}s \\
&= \int_{\Gamma_\mathrm{R}} \left(-\alpha\operatorname{div}_\tau(\phi_2(\mathsf{T}_2 - \mathsf{T}_\mathrm{ext})\boldsymbol{\theta}) - \kappa_2\nabla_\tau\mathsf{T}_2\cdot\nabla_\tau\phi_2(\boldsymbol{\theta}\cdot\boldsymbol{n}) - \alpha\phi_2(\boldsymbol{\theta}\cdot\boldsymbol{n})\frac{\partial}{\partial\boldsymbol{n}}(\mathsf{T}_2 - \mathsf{T}_\mathrm{ext})\right) \mathrm{d}s.
\end{aligned}
$$

Finally, integrating by parts on the surface Γ_R (see [23, Proposition 5.4.9]) yields the desired formula. □

Finally, after the existence of the shape derivative of T is assured, the shape derivative derivative can be computed by using the chain rule.

Proof of Proposition 2.5. Differentiability. Let $\boldsymbol{\theta} \in \Theta_{\rm ad}$. We have

$$J(\Omega_2^{\boldsymbol{\theta}}) = \int_{\Gamma_{\rm R}^{\boldsymbol{\theta}}} \alpha^2 (\mathsf{T}_{\boldsymbol{\theta},2} - \mathsf{T}_{\rm ext})^2 \,\mathrm{d}s.$$

Changing of variables with $\Gamma_{\rm R}^{\boldsymbol{\theta}} = (\boldsymbol{I} + \boldsymbol{\theta})\Gamma_{\rm R}$, we get

$$J(\Omega_2^{\boldsymbol{\theta}}) = \int_{\Gamma_{\rm R}} \alpha^2 (\overline{\mathsf{T}}_{\boldsymbol{\theta},2} - \mathsf{T}_{\rm ext} \circ (\boldsymbol{I} + \boldsymbol{\theta}))^2 B(\boldsymbol{\theta}) \,\mathrm{d}s,$$

where $B(\boldsymbol{\theta})$ is defined in (3.3). We recall that $\boldsymbol{\theta} \in \Theta_{\rm ad} \mapsto B(\boldsymbol{\theta}) \in \mathcal{C}^0(\Gamma_{\rm R})$ and $\boldsymbol{\theta} \in \Theta_{\rm ad} \mapsto \mathsf{T}_{\rm ext} \circ (\boldsymbol{I} + \boldsymbol{\theta}) \in \mathrm{H}^1(\mathbb{R}^d)$ are $\mathcal{C}^1$, and we have previously proved in Proposition 3.1 that $\boldsymbol{\theta} \in \Theta_{\rm ad} \mapsto \overline{\mathsf{T}}_{\boldsymbol{\theta}} \in \mathcal{H}_{\mathsf{T}_{\rm D}}(\Omega_1, \Omega_2)$ is differentiable in a neighborhood of 0. Therefore we deduce by chain rule that the heat insulation J given by (1.5) is shape differentiable and its shape derivative reads:

$$J'(\Omega_2)(\boldsymbol{\theta}) = \int_{\Gamma_{\rm R}} 2\alpha^2 (\mathsf{T}_2 - \mathsf{T}_{\rm ext})(\dot{\mathsf{T}}_2 - \nabla \mathsf{T}_{\rm ext} \cdot \boldsymbol{\theta}) \,\mathrm{d}s + \int_{\Gamma_{\rm R}} \alpha^2 (\mathsf{T}_2 - \mathsf{T}_{\rm ext})^2 \operatorname{div}_\tau(\boldsymbol{\theta}) \,\mathrm{d}s. \tag{3.4}$$

Shape derivative computation. Taking $\phi = \mathsf{R} \in \mathcal{H}_0(\Omega_1, \Omega_2)$ in the material derivative equation (3.1) and testing with $\dot{\mathsf{T}} \in \mathcal{H}_0(\Omega_1, \Omega_2)$ in the adjoint equation (2.7), we get respectively

$$\begin{aligned}
\int_{\Omega_2} \kappa_2 \nabla \dot{\mathsf{T}}_2 \cdot \nabla \mathsf{R}_2 \,\mathrm{d}x &+ \int_{\Omega_1} \left(\kappa_1 \nabla \dot{\mathsf{T}}_1 \cdot \nabla \mathsf{R}_1 + \nabla \dot{\mathsf{T}}_1 \cdot \boldsymbol{u} \mathsf{R}_1 \right) \mathrm{d}x + \int_{\Gamma_{\rm R}} \alpha \dot{\mathsf{T}}_2 \mathsf{R}_2 \;\mathrm{d}s \\
&+ \int_{\Gamma} \kappa_{\rm m} \left(\epsilon \nabla_\tau \langle \mathsf{R} \rangle \cdot \nabla_\tau \left\langle \dot{\mathsf{T}} \right\rangle + H \langle \mathsf{R} \rangle [\dot{\mathsf{T}}] + \frac{1}{\epsilon} [\mathsf{R}][\dot{\mathsf{T}}] \right) \mathrm{d}s \\
&= \int_{\Omega_2} \kappa_2 \left(\nabla \boldsymbol{\theta} + \nabla \boldsymbol{\theta}^t - \operatorname{div}(\boldsymbol{\theta}) \mathrm{I} \right) \nabla \mathsf{T}_2 \cdot \nabla \mathsf{R}_2 \,\mathrm{d}x \\
&\quad - \int_{\Gamma_{\rm R}} \alpha \left(\operatorname{div}_\tau(\boldsymbol{\theta})(\mathsf{T}_2 - \mathsf{T}_{\rm ext}) \mathsf{R}_2 - \nabla \mathsf{T}_{\rm ext} \cdot \boldsymbol{\theta} \mathsf{R}_2 \right) \,\mathrm{d}s,
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
\int_{\Omega_2} \kappa_2 \nabla \dot{\mathsf{T}}_2 \cdot \nabla \mathsf{R}_2 \,\mathrm{d}x &+ \int_{\Omega_1} \left(\kappa_1 \nabla \dot{\mathsf{T}}_1 \cdot \nabla \mathsf{R}_1 + \mathsf{R}_2 \boldsymbol{u} \cdot \nabla \dot{\mathsf{T}}_2 \right) \mathrm{d}x + \int_{\Gamma_{\rm R}} \alpha \dot{\mathsf{T}}_2 \mathsf{R}_2 \,\mathrm{d}s \\
&+ \int_{\Gamma} \kappa_{\rm m} \left(\epsilon \nabla_\tau \langle \mathsf{R} \rangle \cdot \nabla_\tau \left\langle \dot{\mathsf{T}} \right\rangle + H \langle \mathsf{R} \rangle [\dot{\mathsf{T}}] + \frac{1}{\epsilon} [\mathsf{R}][\dot{\mathsf{T}}] \right) \mathrm{d}s = \int_{\Gamma_{\rm R}} 2\alpha^2 (\mathsf{T}_2 - \mathsf{T}_{\rm ext}) \dot{\mathsf{T}}_2 \,\mathrm{d}s.
\end{aligned} \tag{3.6}$$

Using (3.5) and (3.6), we get

$$\begin{aligned}
\int_{\Gamma_{\rm R}} 2\alpha^2 (\mathsf{T}_2 - \mathsf{T}_{\rm ext}) \dot{\mathsf{T}}_2 \,\mathrm{d}s &= \int_{\Omega_2} \kappa_2 \left(\nabla \boldsymbol{\theta} + \nabla \boldsymbol{\theta}^t - \operatorname{div}(\boldsymbol{\theta}) \mathrm{I} \right) \nabla \mathsf{T}_2 \cdot \nabla \mathsf{R}_2 \,\mathrm{d}x \\
&\quad - \int_{\Gamma_{\rm R}} \alpha \left(\operatorname{div}_\tau(\boldsymbol{\theta})(\mathsf{T}_2 - \mathsf{T}_{\rm ext}) \mathsf{R}_2 - \nabla \mathsf{T}_{\rm ext} \cdot \boldsymbol{\theta} \mathsf{R}_2 \right) \,\mathrm{d}s.
\end{aligned} \tag{3.7}$$

Plugging (3.7) into (3.4), this yields:

$$\begin{aligned}
J'(\Omega_2)(\boldsymbol{\theta}) = \int_{\Gamma_{\rm R}} &\left(\operatorname{div}_\tau(\boldsymbol{\theta})(\alpha^2 (\mathsf{T}_2 - \mathsf{T}_{\rm ext}) - \alpha (\mathsf{T}_2 - \mathsf{T}_{\rm ext}) \mathsf{R}_2) - (2\alpha^2 (\mathsf{T}_2 - \mathsf{T}_{\rm ext}) - \alpha \mathsf{R}_2) \nabla \mathsf{T}_{\rm ext} \cdot \boldsymbol{\theta} \right) \,\mathrm{d}s \\
&+ \int_{\Omega_2} \kappa_2 (\nabla \boldsymbol{\theta} + \nabla \boldsymbol{\theta}^t - \operatorname{div}(\boldsymbol{\theta}) \mathrm{I}) \nabla \mathsf{T}_2 \cdot \nabla \mathsf{R}_2 \,\mathrm{d}x,
\end{aligned}$$

obtaining (2.6).

To prove the surface expression (2.8), since now we have more regularity, we can integrate by parts, yielding to the terms $\boldsymbol{\theta}\cdot\boldsymbol{n}$. By chain rule,

$$J'(\Omega_2)(\boldsymbol{\theta}) = \int_{\Gamma_{\mathrm{R}}} 2\alpha^2(\mathsf{T}_2-\mathsf{T}_{\mathrm{ext}})\mathsf{T}_2'\,\mathrm{d}s + \int_{\Gamma_{\mathrm{R}}} \alpha^2\left(\frac{\partial}{\partial\boldsymbol{n}}(\mathsf{T}_2-\mathsf{T}_{\mathrm{ext}})^2 + H(\mathsf{T}_2-\mathsf{T}_{\mathrm{ext}})^2\right)(\boldsymbol{\theta}\cdot\boldsymbol{n})\,\mathrm{d}s. \tag{3.8}$$

Testing the Eulerian derivative equation (2.5) with $\mathsf{R}\in\mathcal{H}_0(\Omega_1,\Omega_2)$ and the adjoint equation (2.7) with $\mathsf{T}'\in\mathcal{H}_0(\Omega_1,\Omega_2)$, we have respectively

$$\begin{aligned}\int_{\Omega_2}\kappa_2\nabla\mathsf{T}_2'\cdot\nabla\mathsf{R}_2\,\mathrm{d}x + \int_{\Omega_1}\left(\kappa_1\nabla\mathsf{T}_1'\cdot\nabla\mathsf{R}_1 + \nabla\mathsf{T}_1'\cdot\boldsymbol{u}\mathsf{R}_1\right)\mathrm{d}x + \int_{\Gamma_{\mathrm{R}}}\alpha\mathsf{T}_2'\mathsf{R}_2\,\mathrm{d}s \\ + \int_{\Gamma}\kappa_{\mathrm{m}}\left(\epsilon\nabla_\tau\langle\mathsf{R}\rangle\cdot\nabla_\tau\langle\mathsf{T}'\rangle + H\langle\mathsf{R}\rangle[\mathsf{T}'] + \frac{1}{\epsilon}[\mathsf{R}][\mathsf{T}']\right)\mathrm{d}s \\ = -\int_{\Gamma_{\mathrm{R}}}\left(\kappa_2\nabla_\tau\mathsf{T}_2\cdot\nabla_\tau\mathsf{R}_2 + \alpha\mathsf{R}_2\left(\frac{\partial}{\partial\boldsymbol{n}}(\mathsf{T}_2-\mathsf{T}_{\mathrm{ext}}) + H(\mathsf{T}_2-\mathsf{T}_{\mathrm{ext}})\right)\right)(\boldsymbol{\theta}\cdot\boldsymbol{n})\,\mathrm{d}s\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\int_{\Omega_2}\kappa_2\nabla\mathsf{T}_2'\cdot\nabla\mathsf{R}_2\,\mathrm{d}x + \int_{\Omega_1}\left(\kappa_1\nabla\mathsf{T}_1'\cdot\nabla\mathsf{R}_1 + \mathsf{R}_1\boldsymbol{u}\cdot\nabla\mathsf{T}_1'\right)\mathrm{d}x + \int_{\Gamma_{\mathrm{R}}}\alpha\mathsf{T}_2'\mathsf{R}_2\,\mathrm{d}s \\ + \int_{\Gamma}\kappa_{\mathrm{m}}\left(\epsilon\nabla_\tau\langle\mathsf{R}\rangle\cdot\nabla_\tau\langle\mathsf{T}'\rangle + H\langle\mathsf{R}\rangle[\mathsf{T}'] + \frac{1}{\epsilon}[\mathsf{R}][\mathsf{T}']\right)\mathrm{d}s = \int_{\Gamma_{\mathrm{R}}} 2\alpha^2(\mathsf{T}_2-\mathsf{T}_{\mathrm{ext}})\mathsf{T}_2'\,\mathrm{d}s.\end{aligned} \tag{3.10}$$

Using (3.9) and (3.10), we get

$$\int_{\Gamma_{\mathrm{R}}} 2\alpha^2(\mathsf{T}_2-\mathsf{T}_{\mathrm{ext}})\mathsf{T}_2'\,\mathrm{d}s = -\int_{\Gamma_{\mathrm{R}}}\left(\kappa_2\nabla_\tau\mathsf{T}_2\cdot\nabla_\tau\mathsf{R}_2 + \alpha\mathsf{R}_2\left(\frac{\partial}{\partial\boldsymbol{n}}(\mathsf{T}_2-\mathsf{T}_{\mathrm{ext}}) + H(\mathsf{T}_2-\mathsf{T}_{\mathrm{ext}})\right)\right)(\boldsymbol{\theta}\cdot\boldsymbol{n})\,\mathrm{d}s. \tag{3.11}$$

Then, (3.8) becomes

$$\begin{aligned}J'(\Omega_2)(\boldsymbol{\theta}) &= \int_{\Gamma_{\mathrm{R}}}\Bigg(\alpha^2\frac{\partial}{\partial\boldsymbol{n}}(\mathsf{T}_2-\mathsf{T}_{\mathrm{ext}})^2 + \alpha^2 H(\mathsf{T}_2-\mathsf{T}_{\mathrm{ext}})^2 - \kappa_2\nabla_\tau\mathsf{T}_2\cdot\nabla_\tau\mathsf{R}_2 \\ &\qquad -\alpha\mathsf{R}_2\left(\frac{\partial}{\partial\boldsymbol{n}}(\mathsf{T}_2-\mathsf{T}_{\mathrm{ext}}) - H(\mathsf{T}_2-\mathsf{T}_{\mathrm{ext}})\right)\Bigg)(\boldsymbol{\theta}\cdot\boldsymbol{n})\,\mathrm{d}s \\ &= \int_{\Gamma_{\mathrm{R}}}\Bigg(2\alpha^2(\mathsf{T}_2-\mathsf{T}_{\mathrm{ext}})\frac{\partial\mathsf{T}_2}{\partial\boldsymbol{n}} + \alpha^2 H(\mathsf{T}_2-\mathsf{T}_{\mathrm{ext}})^2 - \alpha\mathsf{R}_2\frac{\partial\mathsf{T}_2}{\partial\boldsymbol{n}} - \alpha H\mathsf{R}_2(\mathsf{T}_2-\mathsf{T}_{\mathrm{ext}}) \\ &\qquad -\kappa_2\nabla_\tau\mathsf{T}_2\cdot\nabla_\tau\mathsf{R}_2 - \frac{\partial\mathsf{T}_{\mathrm{ext}}}{\partial\boldsymbol{n}}(2\alpha^2(\mathsf{T}_2-\mathsf{T}_{\mathrm{ext}}) - \alpha\mathsf{R}_2)\Bigg)(\boldsymbol{\theta}\cdot\boldsymbol{n})\,\mathrm{d}s.\end{aligned}$$

Using the boundary conditions $\kappa_2\frac{\partial\mathsf{T}_2}{\partial\boldsymbol{n}} = \alpha(\mathsf{T}_{\mathrm{ext}}-\mathsf{T}_2)$ and $\kappa_2\frac{\partial\mathsf{R}_2}{\partial\boldsymbol{n}} = 2\alpha(\mathsf{T}_2-\mathsf{T}_{\mathrm{ext}}) - \alpha\mathsf{R}_2$ on Γ_{R} and that $\nabla_\tau\mathsf{T}_2\cdot\nabla_\tau\mathsf{R}_2 = \nabla\mathsf{T}_2\cdot\nabla\mathsf{R}_2 - \frac{\partial\mathsf{T}_2}{\partial\boldsymbol{n}}\frac{\partial\mathsf{R}_2}{\partial\boldsymbol{n}}$, we obtain (2.8). □

Remark 3.2. *We also expose an alternative method to compute the shape derivative of a general functional by only using the Eulerian derivative in Appendix A.*

3.3 Shape sensitivity analysis with random exterior temperature

Proof of Proposition 2.7. Let $\omega \in \Xi$ be fixed. Recall that $\mathsf{T}_{\text{ext}}(x,\omega) = \mathsf{T}^0_{\text{ext}}(x) + \sum_{k=1}^{m} \xi_k(\omega)\mathsf{T}^k_{\text{ext}}(x)$, for $x \in \Omega$. Then, by linearity,

$$\mathsf{T}(\cdot,\omega) = \mathsf{T}^0(\cdot) + \sum_{k=1}^{m} \xi_k(\omega)\mathsf{T}^k(\cdot)$$

is the unique solution in $\mathcal{H}_{\mathsf{T}_\text{D}}(\Omega_1,\Omega_2)$ of (2.9) where, for each $k = 1,\dots,m$, $\mathsf{T}^k \in \mathcal{H}_0(\Omega_1,\Omega_2)$ solves Equation (1.4) with respectively $\mathsf{T}^k_{\text{ext}}$ and 0 as conditions on Γ_R and Γ_D, and where $\mathsf{T}^0 \in \mathcal{H}_{\mathsf{T}_\text{D}}(\Omega_1,\Omega_2)$ solves Equation (1.4) with respectively $\mathsf{T}^0_{\text{ext}}$ and T_D as conditions on Γ_R and Γ_D.

Now we will show (2.11). Using that J is quadratic with respect to the temperature gap at the boundary, we have

$$\begin{aligned}
\mathbb{E}[J](\Omega_2,\mathsf{T}_{\text{ext}}) &= \mathbb{E}[J(\Omega_2,\mathsf{T}^0_{\text{ext}} + \sum_{k=1}^{m}\xi_k\mathsf{T}^k_{\text{ext}}] = \mathbb{E}\left[\int_{\Gamma_\text{R}} \alpha^2\left((\mathsf{T}^0 - \mathsf{T}^0_{\text{ext}} + \sum_{k=1}^{m}\xi_k(\mathsf{T}^k - \mathsf{T}^k_{\text{ext}})\right)^2\right] \\
&= \int_{\Gamma_\text{R}} \alpha^2(\mathsf{T}^0 - \mathsf{T}^0_{\text{ext}})^2 + 2\sum_{k=1}^{n}\mathbb{E}[\xi_k]\int_{\Gamma_\text{R}}(\mathsf{T}^0 - \mathsf{T}^0_{\text{ext}})(\mathsf{T}^k - \mathsf{T}^k_{\text{ext}}) \\
&\quad + \sum_{k,l=1}^{n}\mathbb{E}[\xi_k\xi_l]\int_{\Gamma}(\mathsf{T}^l - \mathsf{T}^l_{\text{ext}})(\mathsf{T}^k - \mathsf{T}^k_{\text{ext}}).
\end{aligned}$$

Since the random variables ξ_k are independent and centered, many terms cancel and one gets

$$\mathbb{E}[J](\Omega_2,\mathsf{T}_{\text{ext}}) = J(\Omega_2,\mathsf{T}^0_{\text{ext}}) + \int_{\Xi}\sum_{k=1}^{m}\xi_k^2 J(\Omega_2,\mathsf{T}^k_{\text{ext}})\mathbb{P}(\mathrm{d}\omega) = J(\Omega_2,\mathsf{T}^0_{\text{ext}}) + \sum_{k=1}^{m}\sigma_k^2 J(\Omega_2,\mathsf{T}^k_{\text{ext}}).$$

The expression of the shape derivative (2.12) follows as a linear combination of shape derivatives of the form of those in Proposition 2.5. □

4 Numerical methods used to solve the involved problems

4.1 Shape optimization framework

The level-set method. In the context of shape optimization, the level set evolution method was introduced by Allaire *et al.* in [4]. The idea consists in considering a fixed domain D that contains every admissible domain Ω and such that boundaries Γ_D, Γ_N and Γ_e belong to ∂D. In practice, D is a box. This allows to describe Ω by means of a level set function $\phi : D \to \mathbb{R}$ as follows

$$\begin{cases} x \in \Omega & \Longleftrightarrow \quad \phi(x) < 0 \\ x \in \Gamma_\text{R} & \Longleftrightarrow \quad \phi(x) = 0 \\ x \in D \setminus \overline{\Omega} & \Longleftrightarrow \quad \phi(x) > 0. \end{cases}$$

In particular this allows us to track the boundary Γ_R that we aim to optimize. Then, the mesh on D is done based on the level-set ϕ, identifying Γ_R by the zero level set of ϕ. After initialization, at the step n of the shape optimization process, we compute the level set ϕ^n by solving the following equation,

$$\begin{cases} \dfrac{\partial \phi^n}{\partial t} + \boldsymbol{\theta} \cdot \nabla \phi^n = 0, & 0 < t < \tau,\ x \in D \\ \phi^n(0,x) = \phi^{n-1}(x), & x \in D, \end{cases} \tag{4.1}$$

where $\tau > 0$ is the descent step in the shape optimization algorithm and $\boldsymbol{\theta}$ is an appropriate velocity field, such that $\tau \|\boldsymbol{\theta}\|_{\mathrm{L}^\infty(D)^d}$ is of the order of mesh size h. In our applications, we rely on the *null space algorithm* (that we briefly describe below), where $\|\boldsymbol{\theta}\|_{\mathrm{L}^\infty(D)^d}$ is at the mesh size scale h, then we can simply choose $\tau = 1$. Numerically speaking, Equation (4.1) can be computed by ADVECT (see [10]) and the remeshing step by MMG (see [17]). Notice that in our case, we have two level set functions, ϕ_1 and ϕ_2 that describe Ω_1 and Ω_2, respectively. Since Ω_1 is fixed, we will just have to update ϕ_2 for the remeshing.

The velocity field $\boldsymbol{\theta}$ that we will use belongs to $\mathrm{H}^1(D)^d$, such that $\boldsymbol{\theta} = 0$ on ∂D and $\boldsymbol{\theta} = 0$ in $\overline{\Omega}_1$. It is obtained by solving the following extension-regularization problem,

$$\int_D \left(h^2 \nabla \boldsymbol{\theta} : \nabla \psi + \boldsymbol{\theta} \cdot \psi\right) \,\mathrm{d}x = \langle J'(\Gamma_{\mathrm{R}}), \psi \rangle, \quad \forall \psi \in \left\{\psi \in \mathrm{H}^1(D)^d;\ \psi = 0 \text{ on } \partial D \text{ and } \psi = 0 \text{ in } \overline{\Omega}_1\right\}. \tag{4.2}$$

It is important to remark, that by construction, $\boldsymbol{\theta}$ is a descent direction.

Null space optimization method. As constrained optimization algorithm, we rely on the *null space algorithm* introduced in [21] under the implementation of Feppon [19]. This method first decreases the violation of the constraint in order to be feasible, then minimizes the objective function. It is particularly well suited when we start from shapes that does not satisfy the constraints and when numerous constraints are considered.

4.2 Numerical resolution with FEM

We highlight that the approximate convection-diffusion (1.4) and the adjoint equations (2.7) can not be implemented directly due to the use of the broken Sobolev spaces such as $\mathcal{H}_0(\Omega_1, \Omega_2)$. Allaire *et al* proposed a method in [2] to approximate this kind of equations in order to use any finite element software with spaces of continuous functions. However this method involves to duplicate the degrees of freedom, which we do not want for our 3D simulations. Indeed, it becomes too expensive in our context. Domain decomposition methods can be used as well, adapting [25] for example. However, it is not clear how many iterations it can take to converge to no mismatch at the interface, in particular in 3D geometries with a large quantity of vertices at the interface (and we require to solve it a lot of times in the shape optimization procedure). Furthermore, the factor $\frac{1}{\epsilon}$ can lead to poor conditioning of the linear systems and then slow resolution. For all these reasons, we solve these equations directly using the dedicated Nitsche method that we have introduced in our previous work [11]. Its main advantages are the efficiency and the robustness with respect to the small parameter ϵ that naive methods do not provide (see our previous work [11]) for details.

Navier-Stokes equations. Concerning the Navier-Stokes equations (1.3), they can be solved with any finite element software. We rely on FreeFem++ (see [22]) and PETSc (see [5]) for this purpose. Since Ω_1 is fixed, the Navier-Stokes equations (1.3) have to be solved just once. However, it is necessary to interpolate the solution in every step of the shape optimization algorithm, since the mesh that describes Ω_1 will slightly change: indeed the remeshing step MMG concerns the level set function that describes the boundary Γ_{R} but then naturally affects the whole mesh.

Nitsche extended finite element method for a Ventcel transmission problem with discontinuities at the interface For the sake of simplicity, in this part, we suppose $\mathsf{T}_\mathrm{D} = 0$. In the case of the convection-diffusion problem (2.3), we decompose the associated bilinear form $a(\cdot,\cdot)$ into $a(\cdot,\cdot) = b(\cdot,\cdot) + c(\cdot,\cdot)$ where

$$\begin{aligned} b(\mathsf{T},\mathsf{S}) &:= \sum_{i=1}^{2} \int_{\Omega_i} \kappa_i \nabla \mathsf{T}_i \cdot \nabla \mathsf{S}_i \,\mathrm{d}x + \int_{\Omega_1} \mathsf{S}_1 \boldsymbol{u} \cdot \nabla \mathsf{T}_1 \,\mathrm{d}x \\ &\quad + \int_{\Gamma_\mathrm{R}} \alpha \mathsf{T}_2 \mathsf{S}_2 \,\mathrm{d}s + \int_{\Gamma} \kappa_\mathrm{m} \left(\epsilon \nabla_\tau \langle \mathsf{T} \rangle \cdot \nabla_\tau \langle \mathsf{S} \rangle + H[\mathsf{T}] \langle \mathsf{S} \rangle \right) \mathrm{d}s, \\ c(\mathsf{T},\mathsf{S}) &:= \int_\Gamma \frac{\kappa_\mathrm{m}}{\epsilon} [\mathsf{T}][\mathsf{S}] \,\mathrm{d}s. \end{aligned}$$

The term $c(\mathsf{T},\mathsf{S})$ produces poor conditioning when ϵ is small, that is the case since our model comes from an asymptotic development. To deal with this, we consider the Nitsche approach [26] that we previously used in [11] to stabilize our matrix with respect to ϵ, improving the conditioning of the matrix. This approach may seem tricky at first sight, so we give a few intuitions here before introducing the method (we refer to [11] for details).

A first (naive) attempt is to add a positive real value $\eta > 0$ to the denominator of $[\mathsf{T}]/\epsilon$, so that the denominator is greater than ϵ, i.e. $[\mathsf{T}]/(\epsilon+\eta)$. Of course, this is a numerical recipe which lacks mathematical support. In particular, it does not satisfy the transmission condition

$$\left\langle \kappa \frac{\partial \mathsf{T}}{\partial \boldsymbol{n}} \right\rangle = -\frac{\kappa_\mathrm{m}}{\epsilon} [\mathsf{T}] \text{ on } \Gamma. \tag{4.3}$$

Then, the intuition is to find an adequate η. More precisely, the new discrete problem must be such that the associated bilinear form is consistent, coercive and continuous. To guarantee consistency, we weakly impose the transmission condition (4.3). For the bilinear form to be continuous and coercive, we need to choose an appropriate penalization coefficient. We first introduce some notations to formalize this.

Let $\mathcal{T}_h$ be a regular simplicial mesh of Ω and let $\mathcal{F}_h$ be the set of faces of $\mathcal{T}_h$, $\mathcal{F}_{h,\Gamma}$ the set of faces situated on Γ and $\mathcal{T}_{h,\Gamma}$ the set of elements which have one face on Γ. Let h_F be the diameter of the face $F \in \mathcal{F}_{h,\Gamma}$. We consider the polynomial spaces

$$\mathrm{P}_h^1 := \{\mathsf{S}_h \in \mathcal{C}(\Omega_1) \times \mathcal{C}(\Omega_2); \mathsf{S}_h|_K \in \mathbb{P}^1, \forall K \in \mathcal{T}_h\} \qquad \text{and} \qquad \mathrm{P}_{h,0}^1 := \mathrm{P}_h^1 \cap \mathcal{H}_0(\Omega_1,\Omega_2).$$

Then, we define the following mesh-depending bilinear form, for any $\mathsf{T}_h, \mathsf{S}_h \in \mathrm{P}_{h,0}^1$,

$$a_h(\mathsf{T}_h,\mathsf{S}_h) := a(\mathsf{T}_h,\mathsf{S}_h) - \sum_{F \in \mathcal{F}_{h,\Gamma}} \beta_F \left(\left\langle \kappa \frac{\partial \mathsf{T}_h}{\partial \boldsymbol{n}} \right\rangle + \frac{\kappa_\mathrm{m}}{\epsilon}[\mathsf{T}_h], \left\langle \kappa \frac{\partial \mathsf{S}_h}{\partial \boldsymbol{n}} \right\rangle + \frac{\kappa_\mathrm{m}}{\epsilon}[\mathsf{S}_h] \right)_{\mathrm{L}^2(F)},$$

for some $\beta_F > 0$ to be appropriately chosen. In our previous work [11], we enforce coercivity and continuity, leading to

$$a_h(\mathsf{T}_h,\mathsf{S}_h) = a(\mathsf{T}_h,\mathsf{S}_h) - \sum_{F \in \mathcal{F}_{h,\Gamma}} \frac{\gamma \epsilon h_F}{\epsilon + \gamma \kappa_\mathrm{m} h_F} \left(\left\langle \kappa \frac{\partial \mathsf{T}_h}{\partial \boldsymbol{n}} \right\rangle + \frac{\kappa_\mathrm{m}}{\epsilon}[\mathsf{T}_h], \left\langle \kappa \frac{\partial \mathsf{S}_h}{\partial \boldsymbol{n}} \right\rangle + \frac{\kappa_\mathrm{m}}{\epsilon}[\mathsf{S}_h] \right)_{\mathrm{L}^2(F)}.$$

Hence

$$a_h(\mathsf{T}_h,\mathsf{S}_h) = b(\mathsf{T}_h,\mathsf{S}_h) + c_h(\mathsf{T}_h,\mathsf{S}_h),$$

with

$$c_h(\mathsf{T}_h,\mathsf{S}_h) \quad := \quad \sum_{F\in\mathcal{F}_{h,\Gamma}} \int_F \frac{\kappa_{\rm m}}{\epsilon+\gamma\kappa_{\rm m}h_F}[\mathsf{T}_h][\mathsf{S}_h] - \frac{\gamma\epsilon h_F}{\epsilon+\gamma\kappa_{\rm m}h_F}\left\langle \kappa\frac{\partial \mathsf{T}_h}{\partial \boldsymbol{n}}\right\rangle\left\langle \kappa\frac{\partial \mathsf{S}_h}{\partial \boldsymbol{n}}\right\rangle$$
$$-\frac{\gamma\kappa_{\rm m}h_F}{\epsilon+\gamma\kappa_{\rm m}h_F}\left(\left\langle \kappa\frac{\partial \mathsf{T}_h}{\partial \boldsymbol{n}}\right\rangle[\mathsf{S}_h] + \left\langle \kappa\frac{\partial \mathsf{S}_h}{\partial \boldsymbol{n}}\right\rangle[\mathsf{T}_h]\right)\,{\rm d}s,$$

where $\gamma > 0$ is a stabilization parameter, that it is small enough in order to guarantee the coercivity of a_h. Let us remark, that in the decomposition of the new bilinear form a_h, the bilinear form b continues to appear; what it changes is the bilinear form c_h instead of c, whose associated matrix has a better conditioning due to the stabilization.

Remark 4.1. *Note that if $\gamma = 0$, then we recover our original (discrete) formulation with $a(\cdot,\cdot)$ as bilinear form. Moreover, if $\gamma\kappa_{\rm m}h_F$ is larger enough than ϵ, then $\frac{\kappa_{\rm m}}{\epsilon+\gamma\kappa_{\rm m}h_F} \approx \frac{1}{\gamma h_F}$, in which case $\gamma > 0$ cannot be too small. This formally explains the improvement of conditioning.*

Then we consider the following Nitsche problem to approximate the equation (2.3) is

$$\begin{cases} \text{Find } \mathsf{T}_h \in {\rm P}^1_{h,0} \text{ such that} \\ a_h(\mathsf{T}_h,\mathsf{S}_h) = l(\mathsf{S}_h), \quad \forall \mathsf{S}_h \in {\rm P}^1_{h,0}, \end{cases} \tag{4.4}$$

that estimates the continuous solution $\mathsf{T} \in \mathcal{H}_0(\Omega_1,\Omega_2)$ of the convection-diffusion equation (1.4) in the energy sense as it is stated in the next result (the proof is a mere adaptation of [11, Theorem 4.6]).

Theorem 4.2 (Error estimate in energy norm). *Let $\mathsf{T} \in \mathcal{H}_{\mathsf{T}_{\rm D}}(\Omega_1,\Omega_2)$ the solution of the continuous convection-diffusion equation (1.4) and T_h the solution of the (discrete) Nitsche problem (4.4). If in addition $\mathsf{T} \in \mathcal{H}^2(\Omega_1,\Omega_2)$, then for γ sufficiently small, there exists a constant $C > 0$ independent of h and ϵ such that:*

$$|||\mathsf{T}-\mathsf{T}_h|||_h \le Ch\left(\sum_{i=1}^{2}\|\kappa_i^{1/2}\mathsf{T}_i\|^2_{{\rm H}^2(\Omega_i)} + \|(\kappa_{\rm m}\epsilon)^{1/2}\langle \mathsf{T}\rangle\|^2_{{\rm H}^2(\Gamma)} + \sum_{F\in\mathcal{F}_{h,\Gamma}}\frac{\kappa_{\rm m}}{\gamma h_F}\|[\mathsf{T}]\|^2_{{\rm H}^1(F)}\right)^{1/2}, \tag{4.5}$$

where $|||\cdot||| := \left(\|\cdot\|^2_{\mathcal{H}_0(\Omega_1,\Omega_2)} + \sum_{F\in\mathcal{F}_{h,\Gamma}}\frac{1}{\epsilon+\gamma h_F}\|[\cdot]\|^2_{{\rm L}^2(F)}\right)^{1/2}$ *is a mesh-dependent norm on* ${\rm P}^1_h$.

We proceed in a similar way concerning the adjoint equation (2.7). Let $\mathsf{R},\mathsf{S} \in \mathcal{H}_0(\Omega_1,\Omega_2)$. We define $\tilde{a}(\mathsf{R},\mathsf{S}) := a(\mathsf{S},\mathsf{R})$ the bilinear form associated to the adjoint problem (2.7) with right-hand side $\tilde{l}(\mathsf{S}) := \int_{\Gamma_{\rm R}} 2\alpha^2(\mathsf{T}_2-\mathsf{T}_{\rm ext})\mathsf{S}_2\,{\rm d}s$. As previously, the matrix associated to the term $c(\cdot,\cdot)$ has poor conditioning. To stabilize it, we define

$$\tilde{a}_h(\mathsf{R},\mathsf{S}) := \tilde{a}(\mathsf{R},\mathsf{S})$$
$$-\sum_{F\in\mathcal{F}_{h,\Gamma}}\frac{\gamma\epsilon h_F}{\epsilon+\gamma\kappa_{\rm m}h_F}\left(\left\langle\kappa\frac{\partial \mathsf{R}}{\partial\boldsymbol{n}}\right\rangle + \frac{\kappa_{\rm m}}{\epsilon}[\mathsf{R}] + \kappa_{\rm m}H\langle\mathsf{R}\rangle, \left\langle\kappa\frac{\partial \mathsf{S}}{\partial\boldsymbol{n}}\right\rangle + \frac{\kappa_{\rm m}}{\epsilon}[\mathsf{S}] + \kappa_{\rm m}H\langle\mathsf{S}\rangle\right)_{{\rm L}^2(F)}.$$

Therefore, we obtain $\tilde{a}_h(\mathsf{R},\mathsf{S}) = a_h(\mathsf{S},\mathsf{R}) - d_h(\mathsf{R},\mathsf{S})$, where

$$d_h(\mathsf{R},\mathsf{S}) := \sum_{F\in\mathcal{F}_{h,\Gamma}}\frac{\gamma\epsilon h_F}{\epsilon+\gamma\kappa_{\rm m}h_F}\int_F H\langle\mathsf{S}\rangle\left(\left\langle\kappa\frac{\partial \mathsf{R}}{\partial\boldsymbol{n}}\right\rangle + \frac{\kappa_{\rm m}}{\epsilon}[\mathsf{R}]\right)$$

$$+H\left\langle \mathsf{R}\right\rangle \left(\left\langle \kappa\frac{\partial \mathsf{S}}{\partial \boldsymbol{n}}\right\rangle + \frac{\kappa_{\mathrm{m}}}{\epsilon}[\mathsf{S}]\right) + \kappa_{\mathrm{m}}H^2\left\langle \mathsf{R}\right\rangle\left\langle \mathsf{S}\right\rangle \,\mathrm{d}s.$$

Then the Nitsche problem considered to approximate the adjoint equation (2.7) is

$$\begin{cases} \text{Find } \mathsf{R}_h \in \mathrm{P}^1_{h,0} \text{ such that} \\ \tilde{a}_h(\mathsf{R}_h, \mathsf{S}_h) = \tilde{l}(\mathsf{S}_h), \quad \forall \mathsf{S}_h \in \mathrm{P}^1_{h,0}. \end{cases} \tag{4.6}$$

The error estimation is similar to the one given in Theorem 4.2. These equations are solved in sequential with our in-house C++ code.

4.3 Summary: brief description of the algorithm used

To summarize the complete shape optimization procedure, we present below each step with the associated computational code or library we use in Algorithm 1.

Algorithm 1 Level-set mesh evolution method for the heat insulation problem

Require an initial domain $\Omega_1 \cup \Omega_2^0$.
Solve Navier-Stokes equations (2.1) in Ω_1 (once and for all). ▷ FreeFem++
for $n = 0, \ldots, n_{\mathrm{maxiter}}$ **do**
  Current domain $\Omega_1 \cup \Omega_2^n$ represented by the mesh $\mathcal{T}_{\Omega_1\cup\Omega_2^n} \subset \mathcal{T}_D$.
  Solve state equation (4.4) by Nitsche method. ▷ C++ in-house code
  Solve adjoint equation (4.6) by Nitsche method. ▷ C++ in-house code
  Compute the descent directions of each functional by solving (4.2). ▷ FreeFem++
  Compute the deformation field $\boldsymbol{\theta}$. ▷ Null-space algorithm
  Update the level-set function ϕ^{n+1} thanks to (4.1). ▷ mshdist and advect
  Remesh thanks to ϕ^{n+1}. ▷ mmg
end for

5 Numerical examples

We consider the thermal insulation problem in dimension three. We consider the inlet velocity $\boldsymbol{u}_{\mathrm{D}}$ as a parabolic profile with maximum speed at the Γ_{D} centered $(0, y_{\mathrm{c}}, z_{\mathrm{c}})$, equal to 1: in other words, $\boldsymbol{u}_{\mathrm{D}} := \big((r^2 - (y-y_{\mathrm{c}})^2 - (z-z_{\mathrm{c}})^2)/r^2, 0, 0\big)$, where r is the radius of Ω_1 which is fixed to 0.1 in the simulations below. Moreover, we consider $\mathsf{T}_{\mathrm{D}} \equiv 40$ and $\epsilon = 10^{-3}$ in the following examples, and except for the last example, we consider $D = [0,1] \times [0,1] \times [0,1]$. Finally, except for the random outer temperature example of subsection 5.1.2, we take $\mathsf{T}_{\mathrm{ext}} = 0$. Let us conclude these preliminaries by highlighting two points.

- *On the Robin coefficient* α. From [28], we know that for α small (with respect to κ_2), the functional decreases by removing insulation material, meanwhile for α large enough the functional decreases by adding insulation material, which is more intuitive. From a physical point of view, the need to remove material when α is small arises from the fact that, in this case, the convective resistance becomes greater than the conductive resistance (see [6, Chapter 3]). From a (formal) mathematical point of view, we can understand the extreme cases: when $\alpha \to 0$, we get $\kappa_2 \frac{\partial \mathsf{T}_2}{\partial \boldsymbol{n}} = 0$ on Γ_{R}, i.e. there is no influence from the environment (for instance, an external fluid such as air acting on the insulator) and so there is no need to insulate, contrary to the case where $\alpha \to \infty$ for which we get $\mathsf{T}_2 = \mathsf{T}_{\mathrm{ext}}$ on Γ_{R}, i.e. the influence of the environment is very important and then more material is needed to reach equilibrium between the temperature on the insulator and the outside temperature. Hence we will consider this latter case and we summarize the values of the parameters chosen in the following Table 1.

κ_1	$1.5 \cdot 10^{-7}$	m^2s^{-1}
κ_2	10^{-7}	m^2s^{-1}
$\kappa_{\rm m}$	$1.1 \cdot 10^{-4}$	m^2s^{-1}
α	$2 \cdot 10^{-5}$	ms^{-1}
ν	10^{-2}	m^2s^{-1}
ρ	10^3	kgm^{-3}

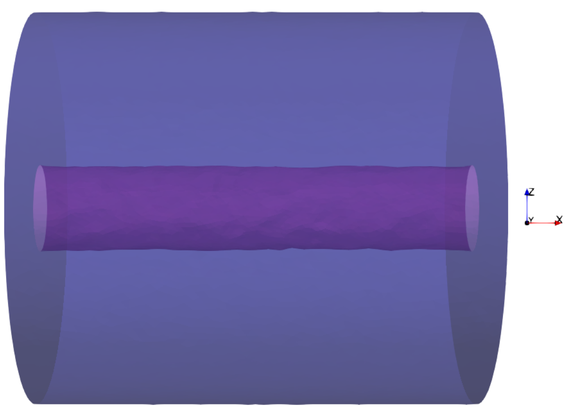
Table 1: Values of the parameters

- *On the no recirculation at the outlet assumption* (2.2). In the four following examples, we numerically check that the hypothesis $\boldsymbol{u} \cdot \boldsymbol{n} \geq 0$ on $\Gamma_{\rm N}$ is well satisfied.

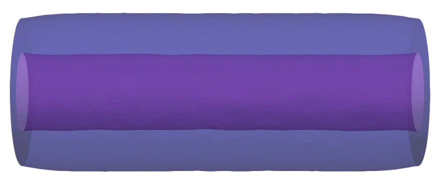
All the presented simulations were performed on a personal laptop with an AMD Ryzen 9 4900hs @3.0 GHz, with 40 GB RAM. The meshes considered vary from 300 to 500 thousand vertices and 2 to 3 million tetrahedra. Each numerical simulation took less than four days of computational time.

5.1 First example: cylinder case

The first example is the cylindric case. We consider here a fixed cylinder Ω_1 of radius $r = 0.1$, of axis (Ox) and with $(y_{\rm c}, z_{\rm c}) = (0.5, 0.5)$. The target volume V_0 is the difference between the volume of a cylinder of radius 0.2 and the volume of Ω_1. It will be chosen similarly in the following examples (changing the cylinder by the respective considered pipes).



(a) Initial domain Ω.



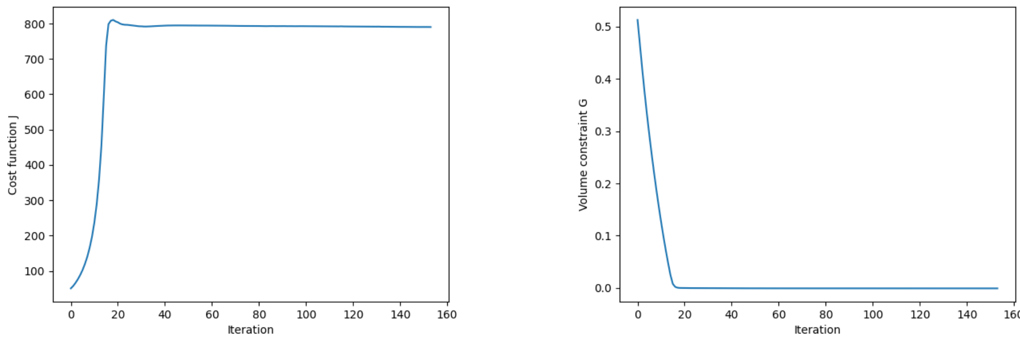
(b) Final domain Ω.

Figure 5: First example - initial and final domains in the deterministic case.

5.1.1 Deterministic case

The initial geometry is depicted on Figure 5a. The optimized design is shown in Figure 5b.

On the one hand, we observe that we do not obtain two concentric cylinders, similarly to what happens in the case of [24]. On the other hand, as expected, the objective functional J increases



(a) Objective function J.

(b) Constraint function G.

Figure 6: First example - convergence history in the deterministic case.

(see Figure 6) until the solution satisfies the volume constraint and then it is optimized (it is well-known and natural that if the volume is larger the insulation is better).

5.1.2 Use of a random outside temperature

We now illustrate, on the previous example, the consideration of a random exterior temperature of the form (following the notations introduced in (2.10))

$$\mathsf{T}_{\text{ext}} = \mathsf{T}^0_{\text{ext}} + \xi_1(\omega)\mathsf{T}^1_{\text{ext}}(x, y, z) + \xi_2(\omega)\mathsf{T}^2_{\text{ext}}(x, y, z),$$

where $\mathsf{T}^0_{\text{ext}} = 0, \mathsf{T}^1_{\text{ext}} = 20x$ and $\mathsf{T}^2_{\text{ext}} = 10z$, the random variables ξ_1, ξ_2 are statistically independent, with zero expectation and variance $\sigma_1^2 = \mathbb{V}\text{ar}(\xi_1) = 0.3$ and $\sigma_2^2 = \mathbb{V}\text{ar}(\xi_2) = 0.7$. Figure 7 summarizes the obtained result, which is very similar to the deterministic case 5. This can be explained as the deterministic component is predominant in the formulas of Theorem 2.7 for the chosen values of the outside temperature, for two reasons. First, since the random part is multiplied by the variance $\sigma_i^2 < 1, i = 1, 2$ and second, since the difference of the values between the outside temperature and inlet Dirichlet temperature is larger for the deterministic case (recall that there are three solutions of the temperature, the deterministic temperature has $\mathsf{T}^0 = 40$ on Γ_{D} and $\mathsf{T}^0_{\text{ext}} = 0$ on Γ_{R} as data, meanwhile the random temperatures have $\mathsf{T}^1 = 0$ on Γ_{D}, $\mathsf{T}^1_{\text{ext}} = 20x$ on Γ_{R} and $\mathsf{T}^2 = 0$ on Γ_{D}, $\mathsf{T}^2_{\text{ext}} = 10z$ on Γ_{R}). The convergence is depicted by Figure 8, where we can actually see that the gap between the deterministic and random case is small with respect to the values of J; the insulation in the random case is slightly larger as expected since it has the random contribution.

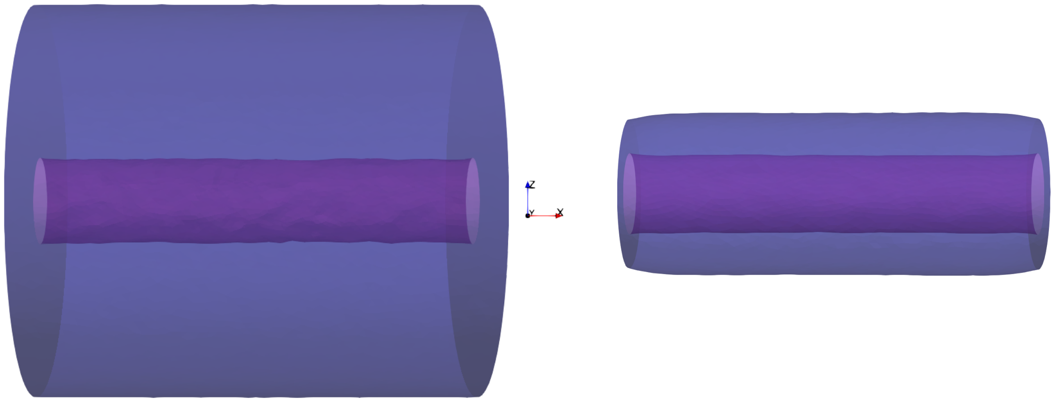
5.2 Second example: perpendicular tubes

In this example, Ω_1 is a pipe of radius $r = 0.1$ with two right-angled bends. The Dirichlet and Neumann boundaries are, respectively,

$$\begin{aligned} \Gamma_{\text{D}} &= \{(0, y, z) \in D;\ (y - 0.5)^2 + (z - 0.75)^2 = r^2\}, \\ \Gamma_{\text{N}} &= \{(1, y, z) \in D;\ (y - 0.5)^2 + (z - 0.25)^2 = r^2\}. \end{aligned}$$

The initial geometry is depicted on Figure 9a.

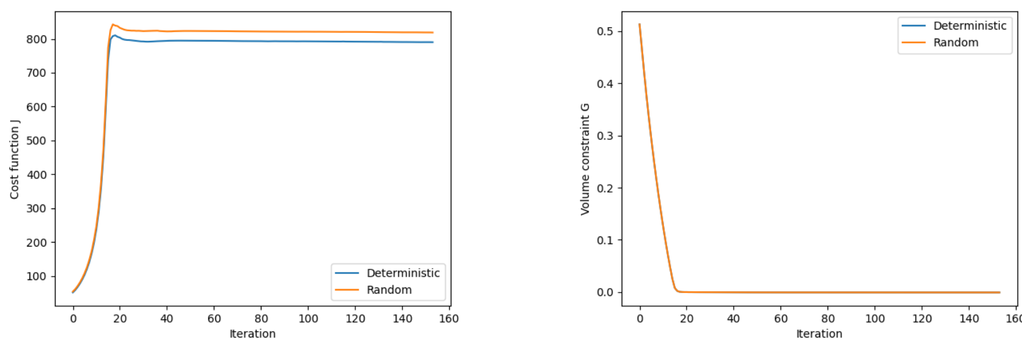
The optimazed design is shown in Figure 9b. In the optimazed domain, the solution satisfies the volume constraint and also keeps better the temperature inside the pipe: this is validated



(a) Initial domain Ω.

(b) Final domain Ω.

Figure 7: First example - initial and final domains with random temperature.



(a) Objective function J.

(b) Constraint function G.

Figure 8: First example - convergence history with random temperature.

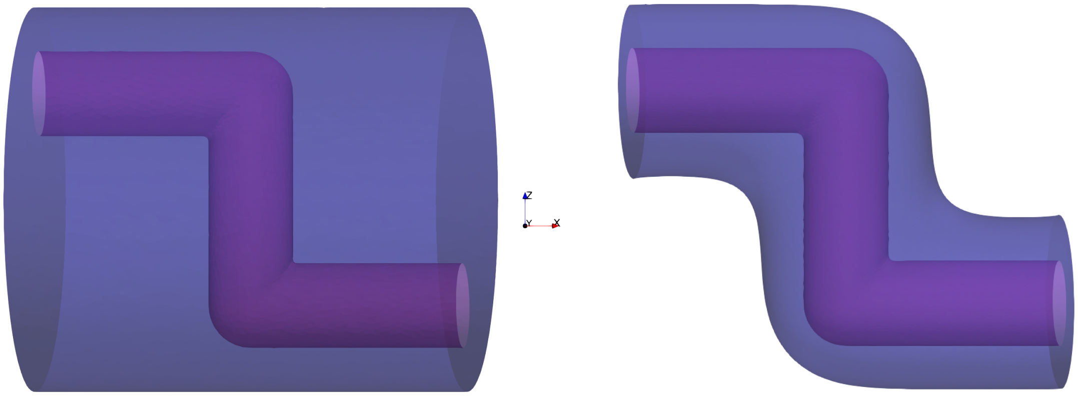
in Figure 10 that shows the convergence history which illustrates that in the first 20 iterations the algorithm tries to satisfy the constraint, decreasing the volume but increasing the objective function J until that the constraint is satisfied, and then the objective function decreases.

5.3 Third example: tubes with angle of inclination

As a third example, we consider a slight variation to the second one, now with an angle of inclination. In this example, the angle formed by the tube at the bottom and at the middle is of $\frac{11}{6}\pi$. The results are similar to the previous case as depicted in Figures 11 and 12.

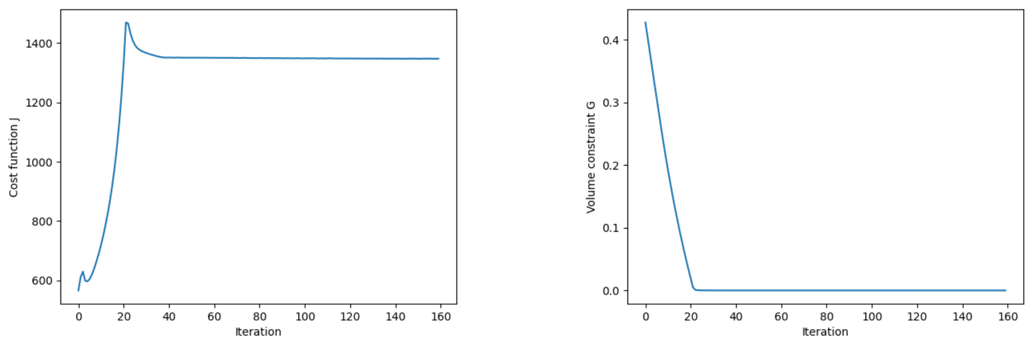
5.4 Fourth example: Z pipe

We conclude these numerical experiments with a Z pipe geometry as Ω_1. Here we consider $D = [0, 2] \times [0, 1] \times [0, 1]$. Figure 13 shows that as in the previous example, the material in the zones far from the pipe is removed. The convergence history is depicted in Figure 14, where we



(a) Initial domain Ω. (b) Final domain Ω.

Figure 9: Second example - initial and final domains.



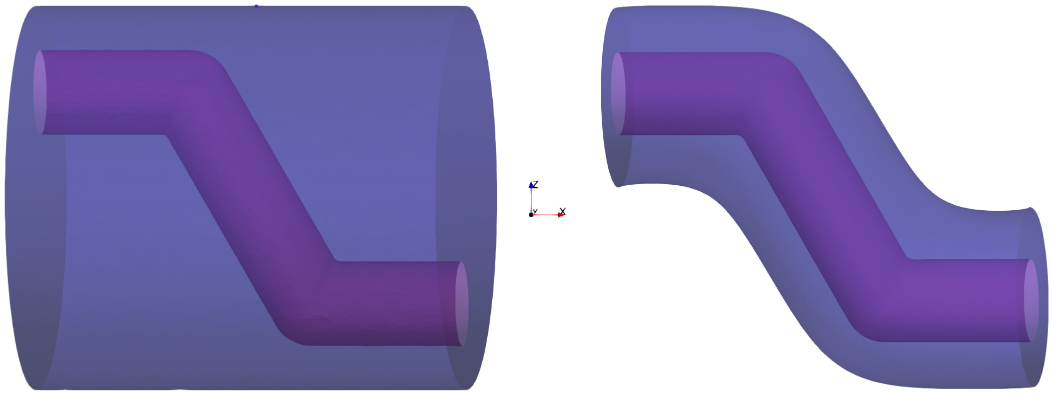
(a) Objective function J. (b) Constraint function G.

Figure 10: Second example - convergence history.

can notice that more iterations are needed to converge (nearly sixty iterations) due to the fact that the shape is larger (in the z axis).

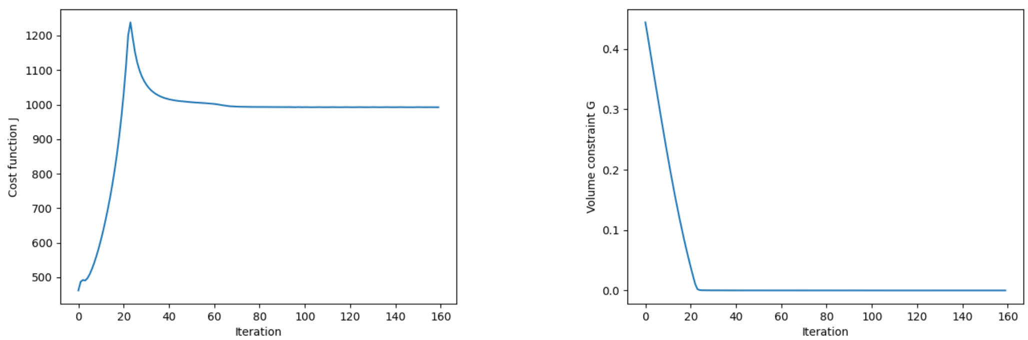
Acknowledgement

Rodrigo Zelada was partially funded by the National Agency for Research and Development (ANID) Scholarship Program BECA DOCTORADO NACIONAL 2021-21211704. Fabien Caubet and Marc Dambrine acknowledge partial support by the ANR project STOIQUES ANR-24-CE40-2216 and by the project RODAM funded by the ISite E2S. Marc Dambrine acknowledge partial support of ARN project SOS2ID ANR-24-CE40-3786. Carlos Conca acknowledges partial support by CMM FB210005 and CeBiB AFB240001 Basal funds for centers of excellence from ANID-Chile, and by Fondecyt project #1250707.



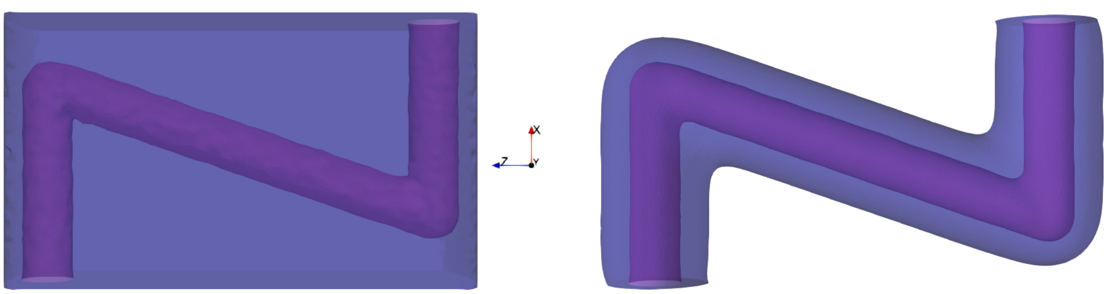
(a) Initial domain Ω. (b) Final domain Ω.

Figure 11: Third example - initial and final domains.



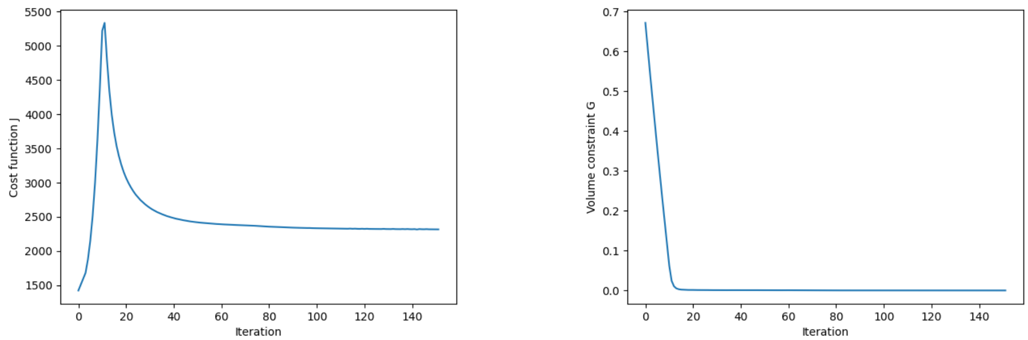
(a) Objective function J. (b) Constraint function G.

Figure 12: Third example - convergence history.



(a) Initial domain Ω. (b) Final domain Ω.

Figure 13: Fourth example - initial and final domains (with $\pi/2$ rotation).



(a) Objective function J.

(b) Constraint function G.

Figure 14: Fourth example - convergence history.

A Shape derivatives using a fully Lagrangian approach

In the present article, we focus on a specific objective functional J measuring the heat loss given in (1.5). In this appendix, we give some results in order to consider another objective functional which will imply to do some computations (the chain rule part) which can be annoying. In [20] was developed a framework to compute the shape derivatives of general functionals for a multi-physics problem, that only requires to compute some partial derivatives.

We keep the same notation as before (see Section 2.2) and consider a general functional J that depends on Ω_2 and on the solution T of the convection-diffusion problem (1.4). We first recall the concept of *transported functional* given in the following definition.

Definition A.1. *The transported functional of J is the functional $\mathcal{J}$ such that for all $\boldsymbol{\theta} \in \Theta_{\mathrm{ad}}$ and all $\hat{\mathsf{T}} \in \mathcal{H}^1(\Omega_1, \Omega_2)$,*

$$\mathcal{J}(\boldsymbol{\theta}, \hat{\mathsf{T}}) := J(\Omega_2^{\boldsymbol{\theta}}, \hat{\mathsf{T}} \circ (\boldsymbol{I} + \boldsymbol{\theta})^{-1}),$$

where $\Omega_2^{\boldsymbol{\theta}} = (\boldsymbol{I} + \boldsymbol{\theta})\Omega_2$.

We suppose that J has continuous partial derivatives at $(\boldsymbol{\theta}, \hat{\mathsf{T}}) = (0, \mathsf{T}(\Omega_2))$. To keep notations as simple as possible, we will omit the evaluations of the partial derivatives at $(\boldsymbol{\theta}, \hat{\mathsf{T}}) = (0, \mathsf{T}(\Omega_2))$. We introduce the solution $\mathsf{R} \in \mathcal{H}_0(\Omega_1, \Omega_2)$ of the following adjoint problem

$$\left\{ \begin{array}{l} \text{Find } \mathsf{R} \in \mathcal{H}_0(\Omega_1, \Omega_2), \text{ such that, for all } \mathsf{S} \in \mathcal{H}_0(\Omega_1, \Omega_2), \\ \displaystyle\sum_{i=1}^{2} \int_{\Omega_i} \kappa_i \nabla \mathsf{R}_i \cdot \nabla \mathsf{S}_i \, \mathrm{d}x + \int_{\Omega_1} \mathsf{R}_1 u \cdot \nabla \mathsf{S}_1 \, \mathrm{d}x + \int_{\Gamma_{\mathrm{R}}} \alpha \mathsf{R}_2 \mathsf{S}_2 \, \mathrm{d}s \\ \displaystyle + \int_{\Gamma} \kappa_{\mathrm{m}} \left(\epsilon \nabla_\tau \langle \mathsf{R} \rangle \cdot \nabla_\tau \langle S \rangle + H \langle \mathsf{R} \rangle [S] + \epsilon^{-1} [\mathsf{R}][S] \right) \mathrm{d}s = \frac{\partial \mathcal{J}}{\partial \hat{\mathsf{T}}}(\mathsf{S}). \end{array} \right. \tag{A.1}$$

Remark A.2. *In the particular case of the insulation functional* (1.5)*,*

$$\frac{\partial \mathcal{J}}{\partial \hat{\mathsf{T}}}(\mathsf{S}) \quad = \quad \int_{\Gamma_{\mathrm{R}}} 2\alpha^2 (\mathsf{T}_2 - \mathsf{T}_{\mathrm{ext}}) \mathsf{S}_2 \, \mathrm{d}s.$$

Then we can give a formula to compute the shape derivative for any functional, requiring just to compute the partial derivative $\frac{\partial \mathcal{J}}{\partial \boldsymbol{\theta}}(\boldsymbol{\theta})$.

Proposition A.3 (Volume shape derivative)**.** *If* $\mathsf{T}_{\text{ext}} \in \mathrm{H}^2(\mathbb{R}^d)$*, then* J *is shape differentiable and the volume shape derivative is given by*

$$J'(\Omega_2)(\boldsymbol{\theta}) = \frac{\partial \mathcal{J}}{\partial \boldsymbol{\theta}}(\boldsymbol{\theta}) + \int_{\Omega_2} \kappa_2 \left(\nabla\boldsymbol{\theta} + \nabla\boldsymbol{\theta}^t - \operatorname{div}(\boldsymbol{\theta})\mathrm{I}\right) \nabla\mathsf{T}_2 \cdot \nabla\mathsf{R}_2 \, \mathrm{d}x \\ - \int_{\Gamma_{\mathrm{R}}} \alpha \left(\operatorname{div}_\tau(\boldsymbol{\theta})(\mathsf{T}_2 - \mathsf{T}_{\text{ext}}) - \nabla\mathsf{T}_{\text{ext}} \cdot \boldsymbol{\theta}\right) \mathsf{R}_2 \, \mathrm{d}s,$$

where $\mathsf{T} \in \mathcal{H}_{\mathsf{T}_{\mathrm{D}}}(\Omega_1, \Omega_2)$ *solves the convection-diffusion equation* (1.4) *and* $\mathsf{R} \in \mathcal{H}_0(\Omega_1, \Omega_2)$ *solves the adjoint equation* (A.1)*.*

Proof. Let $\dot{\mathsf{T}} \in \mathcal{H}_0(\Omega_1, \Omega_2)$ the Lagrangian derivative of $\mathsf{T} \in \mathcal{H}_{\mathsf{T}_{\mathrm{D}}}(\Omega_1, \Omega_2)$ given in (3.1). We obtain the result by using the chain rule

$$J'(\Omega_2)(\boldsymbol{\theta}) = \frac{\partial \mathcal{J}}{\partial \boldsymbol{\theta}}(\boldsymbol{\theta}) + \frac{\partial \mathcal{J}}{\partial \hat{\mathsf{T}}}(\dot{\mathsf{T}})$$

and then proceeding as in the proof of Proposition 2.5. □

Remark A.4. *In the particular case of the insulation functional* (1.5)*,*

$$\frac{\partial \mathcal{J}}{\partial \boldsymbol{\theta}}(\boldsymbol{\theta}) = \int_{\Gamma_{\mathrm{R}}} \alpha^2 \left(\operatorname{div}_\tau(\boldsymbol{\theta})(\mathsf{T}_2 - \mathsf{T}_{\text{ext}})^2 - 2(\mathsf{T}_2 - \mathsf{T}_{\text{ext}})(\nabla\mathsf{T}_{\text{ext}} \cdot \boldsymbol{\theta})\right) \mathrm{d}s. \tag{A.2}$$

The proof is analogous to Proposition 2.4. More precisely, changing variables in $\Omega_2^{\boldsymbol{\theta}} = (\boldsymbol{I} + \boldsymbol{\theta})\Omega_2$ *and differentiating.*

We can also get a surface expression, which is an integral over the free boundary, that in this case is Γ_{R}, and depending only on the normal component of the perturbation field. The result is obtained by integrating by parts the previous formula and using the structure theorem (see [23, Proposition 5.9.1, Theorem 5.9.2] or [27, Theorem 2.27]).

Proposition A.5 (Surface shape derivative)**.** *Under the same hypothesis of the previous proposition and if in addition* $\mathsf{T}_2, \mathsf{R}_2 \in \mathrm{H}^2(\Omega_2)$*, then the shape derivative is given by*

$$J'(\Omega_2)(\boldsymbol{\theta}) = \overline{\frac{\partial \mathcal{J}}{\partial \boldsymbol{\theta}}}(\boldsymbol{\theta}) + \int_{\Gamma_{\mathrm{R}}} \left(\kappa_2 \frac{\partial \mathsf{T}_2}{\partial \boldsymbol{n}} \frac{\partial \mathsf{R}_2}{\partial \boldsymbol{n}} - \kappa_2 \nabla_\tau \mathsf{T}_2 \cdot \nabla_\tau \mathsf{R}_2 - H\alpha(\mathsf{T}_2 - \mathsf{T}_{\text{ext}})\mathsf{R}_2 + \alpha \frac{\partial \mathsf{T}_{\text{ext}}}{\partial \boldsymbol{n}} \mathsf{R}_2\right) (\boldsymbol{\theta} \cdot \boldsymbol{n}) \, \mathrm{d}s, \tag{A.3}$$

where $\overline{\frac{\partial \mathcal{J}}{\partial \boldsymbol{\theta}}}$ *is the part of* $\frac{\partial \mathcal{J}}{\partial \boldsymbol{\theta}}$ *that depends only on* $\boldsymbol{\theta} \cdot \boldsymbol{n}$*.*

Remark A.6. *In the particular case of the insulation functional* (1.5)*,*

$$\overline{\frac{\partial \mathcal{J}}{\partial \boldsymbol{\theta}}}(\boldsymbol{\theta}) = \int_{\Gamma_{\mathrm{R}}} \left(H\alpha^2(\mathsf{T}_2 - \mathsf{T}_{\text{ext}})^2 - 2\alpha^2(\mathsf{T}_2 - \mathsf{T}_{\text{ext}}) \frac{\partial \mathsf{T}_{\text{ext}}}{\partial \boldsymbol{n}}\right) (\boldsymbol{\theta} \cdot \boldsymbol{n}) \, \mathrm{d}s.$$

which is obtained integrating by parts in formula (A.2) *and taking the normal component.*

References

[1] G. Allaire. *Conception optimale de structures*, volume 58 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 2007. With the collaboration of Marc Schoenauer (INRIA) in the writing of Chapter 8.

[2] G. Allaire, B. Bogosel, and M. Godoy. Shape optimization of an imperfect interface: steady-state heat diffusion. *Journal of Optimization Theory and Applications*, 191(1):169–201, 2021.

[3] G. Allaire and C. Dapogny. A deterministic approximation method in shape optimization under random uncertainties. *SMAI J. Comput. Math.*, 1:83–143, 2015.

[4] G. Allaire, C. Dapogny, and P. Frey. A mesh evolution algorithm based on the level set method for geometry and topology optimization. *Structural and Multidisciplinary Optimization*, 48(4):711–715, 2013.

[5] S. Balay, S. Abhyankar, M. F. Adams, J. Brown, P. Brune, K. Buschelman, L. Dalcin, A. Dener, V. Eijkhout, W. D. Gropp, D. Kaushik, M. G. Knepley, D. A. May, L. C. McInnes, R. T. Mills, T. Munson, K. Rupp, P. Sanan, B. F. Smith, S. Zampini, H. Zhang, and H. Zhang. PETSc users manual. Technical Report ANL-95/11 - Revision 3.11, Argonne National Laboratory, 2019.

[6] T. L. Bergman, A. S. Lavine, F. P. Incropera, and D. P. DeWitt. *Introduction to heat transfer*. John Wiley & Sons, 2011.

[7] F. Boyer and P. Fabrie. *Mathematical tools for the study of the incompressible Navier-Stokes equations and related models*, volume 183 of *Applied Mathematical Sciences*. Springer, New York, 2013.

[8] D. Bucur, G. Buttazzo, and C. Nitsch. Two optimization problems in thermal insulation. *Notices of the American Mathematical Society*, 64(8):830–835, 2017.

[9] D. Bucur, M. Nahon, C. Nitsch, and C. Trombetti. Shape optimization of a thermal insulation problem. *Calculus of Variations and Partial Differential Equations*, 61(5):Paper No. 186, 29, 2022.

[10] C. Bui, C. Dapogny, and P. Frey. An accurate anisotropic adaptation method for solving the level set advection equation. *International Journal for Numerical Methods in Fluids*, 70(7):899–922, 2012.

[11] D. Capatina, F. Caubet, M. Dambrine, and R. Zelada. Nitsche extended finite element method of a Ventcel transmission problem with discontinuities at the interface. *ESAIM: Mathematical Modelling and Numerical Analysis*, 2025.

[12] F. Caubet, C. Conca, M. Dambrine, and R. Zelada. Shape optimization for heat exchangers with a thin layer. In *Sixteenth International Conference Zaragoza-Pau on Mathematics and its Applications*, volume 43, pages 51–61. Prensas de la Universidad de Zaragoza, 2024.

[13] F. Caubet, M. Dambrine, G. Gargantini, and J. Maynadier. Shape Optimization of Polynomial Functionals under Uncertainties on the Right-Hand Side of the State Equation. preprint hal-04082741, Jan. 2023.

[14] F. Caubet, H. Haddar, J.-R. Li, and D. V. Nguyen. New transmission condition accounting for diffusion anisotropy in thin layers applied to diffusion MRI. *ESAIM. Mathematical Modelling and Numerical Analysis*, 51(4):1279–1301, 2017.

[15] M. Dambrine, C. Dapogny, and H. Harbrecht. Shape optimization for quadratic functionals and states with random right-hand sides. *SIAM Journal on Control and Optimization*, 53(5):3081–3103, 2015.

[16] M. Dambrine, H. Harbrecht, and B. Puig. Incorporating knowledge on the measurement noise in electrical impedance tomography. *ESAIM. Control, Optimisation and Calculus of Variations*, 25:Paper No. 84, 16, 2019.

[17] C. Dapogny, C. Dobrzynski, and P. Frey. Three-dimensional adaptive domain remeshing, implicit domain meshing, and applications to free and moving boundary problems. *Journal of Computational Physics*, 262:358–378, 2014.

[18] B. Engquist and J.-C. Nédélec. Effective boundary conditions for electromagnetic scattering in thin layers. Technical Report 278, CMAP, 1993.

[19] F. Feppon. Null space optimizer, 2019. `https://gitlab.com/florian.feppon/null-space-optimizer` [Accessed: 21/10/2024].

[20] F. Feppon, G. Allaire, F. Bordeu, J. Cortial, and C. Dapogny. Shape optimization of a coupled thermal fluid-structure problem in a level set mesh evolution framework. *SeMA Journal. Boletin de la Sociedad Española de Matemática Aplicada*, 76(3):413–458, 2019.

[21] F. Feppon, G. Allaire, and C. Dapogny. Null space gradient flows for constrained optimization with applications to shape optimization. *ESAIM. Control, Optimisation and Calculus of Variations*, 26:Paper No. 90, 45, 2020.

[22] F. Hecht. New development in freefem++. *Journal of Numerical Mathematics*, 20(3-4):251–265, 2012.

[23] A. Henrot and M. Pierre. *Variation et optimisation de formes*, volume 48 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer, Berlin, 2005. Une analyse géométrique. [A geometric analysis].

[24] A. Henrot and Y. Privat. What is the optimal shape of a pipe? *Archive for Rational Mechanics and Analysis*, 196(1):281–302, 2010.

[25] G. McBain. Domain decomposition techniques for interfacial discontinuities. In *Fourth FreeFem workshop on Generic Solver for PDEs*, Paris, December 2012. Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie.

[26] J. Nitsche. Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 36:9–15, 1971.

[27] J. Sokołowski and J.-P. Zolésio. *Introduction to shape optimization*, volume 16 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1992. Shape sensitivity analysis.

[28] S. Tozza and G. Toraldo. Numerical hints for insulation problems. *Applied Mathematics Letters. An International Journal of Rapid Publication*, 123:Paper No. 107609, 8, 2022.

[29] G. Vial. *Analyse multi-échelle et conditions aux limites approchées pour un probleme avec couche mince dans un domaine coin.* PhD thesis, Université de Rennes 1, 2003.